Test 2: Differential Equations

Math 341 Wednesday April 27, 2005

Name: ______________________________

Directions: Read *ALL* three (3) problems first before answering any of them. This is a one hour, open notes, open book, test. This test has 7 pages. You must show all relevant work to support your answers. Use complete English sentences and indicate your final answer from your "scratch work."

No.	Score	Maximum
1		30
2		40
3		30
BONUS		10
Total		**100**

1. *[30 points total.]* **Linear Systems of Equations.**

Consider the homogeneous system, $\frac{d\vec{x}}{dt} = A\vec{x}$, where $A = \begin{bmatrix} 12 & -9 \\ 4 & 0 \end{bmatrix}$.

(a) *[10 points].* Find the eigenvalues of A and their associated eigenvectors.

(b) *[10 points].* Write down the general form of the solution $\vec{x}(t)$.

(c) *[10 points].* Find the solution when $\vec{x}(0) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$.

2. *[40 points total.]* **Laplace Transforms, Bessel Functions**
Our goal in this problem is to obtain the Laplace Transform of a Bessel Function. Consider the initial value problem in **Question 60, Page 314**,

$$ty'' + y' + ty = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

(a) *[5 points].* **Confirm** that the exact solution of this initial value problem is Bessel's function of order zero of the first kind $y(t) = J_0(t)$.

(b) *[15 points].* Given that $\mathcal{L}[ty'] = -\frac{d}{ds}\{\mathcal{L}[y']\} = -\frac{d}{ds}\{sY(s) - y(0)\} = -s\frac{dY}{ds} - Y$, show that applying the Laplace Transform to the given initial value problem produces the equation $\mathcal{L}[ty'' + y' + ty] = -(s^2 + 1)\frac{dY}{ds} - sY(s) = 0$.

(c) *[10 points]*. Solve the differential equation in **(b)** for $Y(s)$. You should have an unknown constant $A = Y(0)$ in your answer.

(d) *[10 points]*. Since $y(t) = J_0(t)$ is a continuous function of exponential order, it is true that $\lim_{s\to\infty} sY(s) = y(0)$. Use this information to obtain the value of A and show that $\mathcal{L}[J_0(t)] = Y(s) = \dfrac{1}{\sqrt{s^2+1}}$.

3. *[30 pts. total]* **Power Series.**

When λ is a known parameter, we have the Laguerre Equation

$$xy'' + (1 - x)y' + \lambda y = 0$$

(a) *5 points.* Show that $x = 0$ is a regular singular point of the Laguerre Equation.

(b) *5 points.* Find and solve the indicial equation of the Laguerre Equation.

(c) *10 points.* Show that the recurrence relation for one of the solutions of this differential equation is $a_n = \frac{(n - 1 - \lambda)}{n^2} a_{n-1}$ for $n \geq 1$.

(d) *10 points.* Show that if $\lambda = n$ is a positive integer (say $n = 3$, for example) then all terms past x^n in the power series expansion of the solution $y(x)$ are zero, and thus the Laguerre differential equation has as its solution an n^{th} degree polynomial, known as **a Laguerre Polynomial** $y(x) = a_0 L_n(x)$. Write down $L_0(x)$, $L_1(x)$, $L_2(x)$ and $L_3(x)$.

BONUS QUESTION *[10 points total.]* **TRUE or FALSE.**
Are the following statements TRUE or FALSE – put your answer in the box. To receive ANY credit, you must also give a brief, and correct, explanation in support of your answer! For example, if you think the answer is **FALSE** providing a counterexample for which the statement is NOT TRUE is best. If you think the answer is **TRUE** you should prove why you think the statement is always true. Your explanation of your answer is worth FOUR TIMES as much as the answer you put in the box.

(a) Recall that $\mathcal{L}[e^{At}] = F(s) = (sI - A)^{-1}$. Suppose A is an $n \times n$ square matrix of real numbers with spectral radius (i.e. absolute value of the largest eigenvalue) $\lambda^* \leq 1$.
TRUE or FALSE: $F(s)$ is defined for all $s > 0$.

(b) **TRUE or FALSE:** $\mathcal{L}[f(t)g(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)]$
