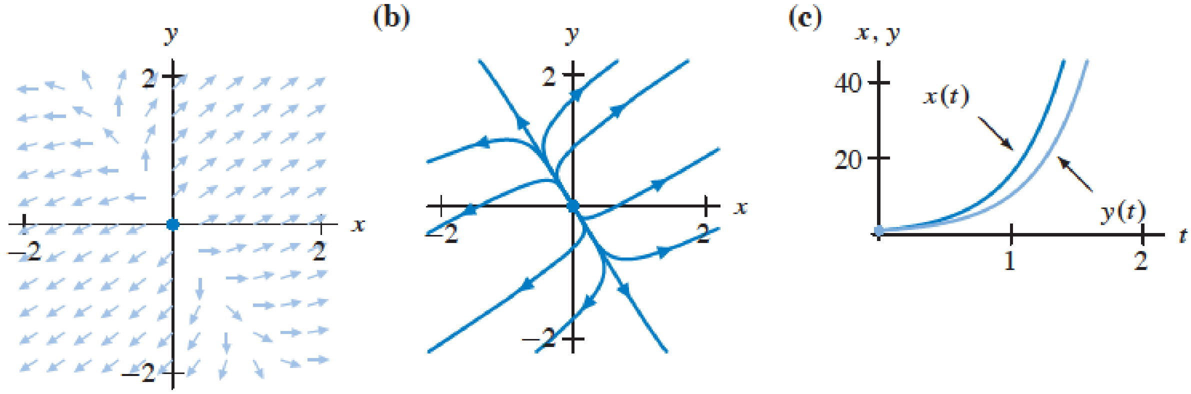
6. $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 3 \\ -0.3 & 3\pi \end{pmatrix} \mathbf{Y}$

7. $\mathbf{Y} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \quad \dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & -2 & -7 \\ -2 & 0 & 6 \\ 0 & 7.3 & 2 \end{pmatrix} \mathbf{Y}$

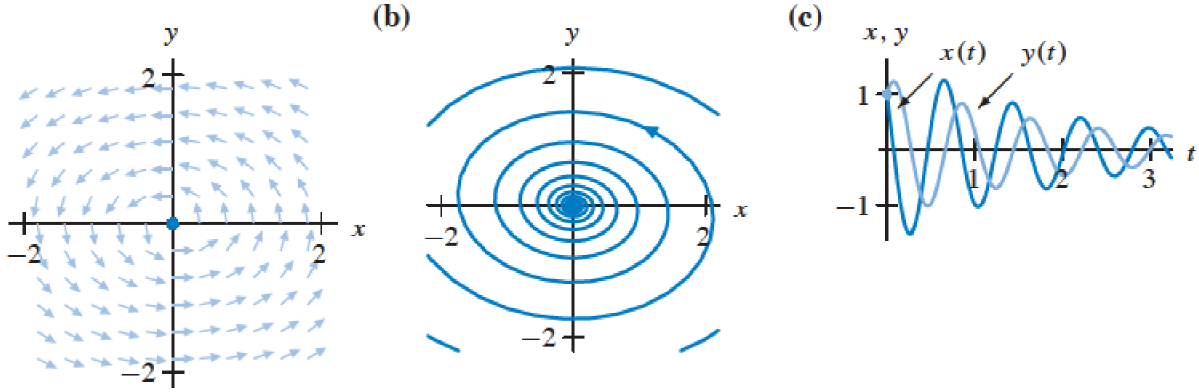
8.

$$\frac{dx}{dt} = -3x + 2\pi y$$
$$\frac{dy}{dt} = 4x - y$$

10. (a) (b) (c)



13. (a) (b) (c)



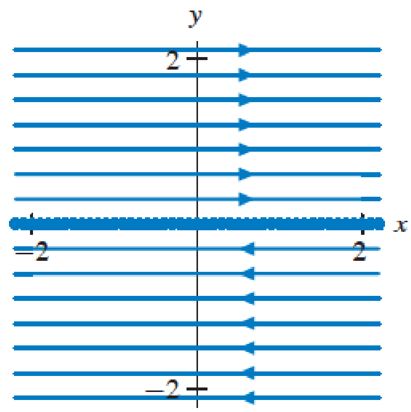
18. In this case, $dv/dt = d^2y/dt^2 = 0$, and the first-order system reduces to

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = 0.$$

(a) Since $dv/dt = 0$, we know that $v(t) = c$ for some constant c.

(b) Since $dy/dt = v = c$, we can integrate to obtain $y(t) = ct + k$ where k is another arbitrary constant. Hence, the general solution of the system consists of all functions of the form $(y(t), v(t)) = (ct + k, c)$ for arbitrary constants c and k.

(c)



8. **(a)** The characteristic polynomial is

$$(2-\lambda)(1-\lambda)-1=\lambda^2-3\lambda+1=0,$$

and therefore the eigenvalues are

$$\lambda_1=\frac{3+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2=\frac{3-\sqrt{5}}{2}.$$

(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = (3+\sqrt{5})/2$, we solve the system of equations

$$\begin{cases} 2x_1 - y_1 = \dfrac{3+\sqrt{5}}{2}x_1 \\ -x_1 + y_1 = \dfrac{3+\sqrt{5}}{2}y_1 \end{cases}$$

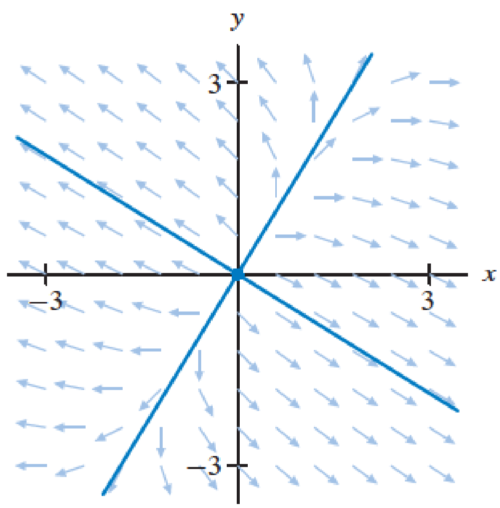
and obtain

$$y_1 = \frac{1-\sqrt{5}}{2}x_1,$$

which is equivalent to the equation $2y_1 = (1-\sqrt{5})x_1$.

Using the same procedure, we obtain the eigenvectors (x_2, y_2) where $2y_2 = (1+\sqrt{5})x_2$ for $\lambda_2 = (3-\sqrt{5})/2$.

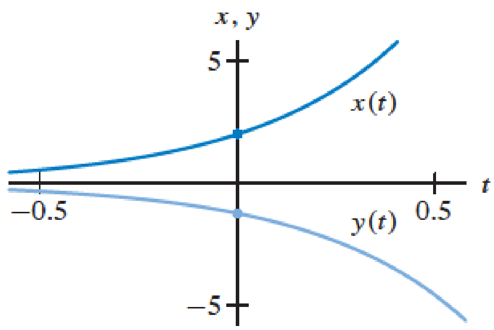
(c)



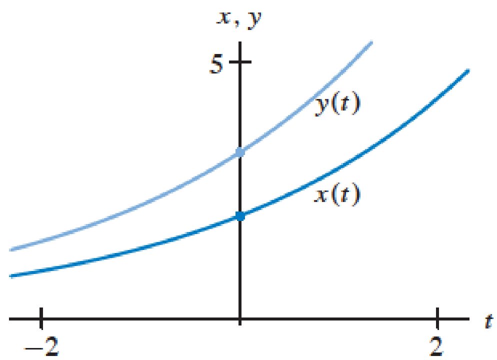
(d) One eigenvector $\mathbf{V}_1$ for the eigenvalue λ_1 is $\mathbf{V}_1 = (2, 1 - \sqrt{5})$, and one eigenvector $\mathbf{V}_2$ for the eigenvalue λ_2 is $\mathbf{V}_2 = (2, 1 + \sqrt{5})$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{(3+\sqrt{5})t/2}\begin{pmatrix} 2 \\ 1 - \sqrt{5} \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{(3-\sqrt{5})t/2}\begin{pmatrix} 2 \\ 1 + \sqrt{5} \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{(3+\sqrt{5})t/2}\begin{pmatrix} 2 \\ 1 - \sqrt{5} \end{pmatrix} + k_2 e^{(3-\sqrt{5})t/2}\begin{pmatrix} 2 \\ 1 + \sqrt{5} \end{pmatrix}.$$

9. **(a)** The characteristic polynomial is

$$(2-\lambda)(1-\lambda)-1=\lambda^2-3\lambda+1=0,$$

and therefore the eigenvalues are

$$\lambda_1=\frac{3+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2=\frac{3-\sqrt{5}}{2}.$$

(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1=(3+\sqrt{5})/2$, we solve the system of equations

$$\begin{cases} 2x_1+y_1=\dfrac{3+\sqrt{5}}{2}x_1 \\ x_1+y_1=\dfrac{3+\sqrt{5}}{2}y_1 \end{cases}$$

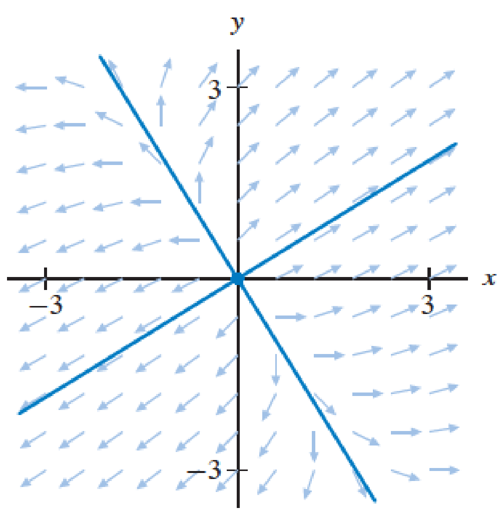
and obtain

$$y_1=\frac{-1+\sqrt{5}}{2}x_1,$$

which is equivalent to the equation $2y_1=(-1+\sqrt{5})x_1$.

Using the same procedure, we obtain the eigenvectors (x_2, y_2) where $2y_2=(-1-\sqrt{5})x_2$ for $\lambda_2=(3-\sqrt{5})/2$.

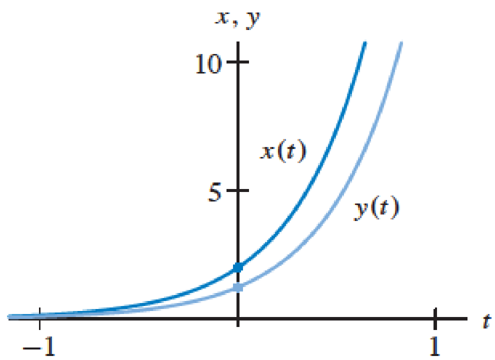
(c)



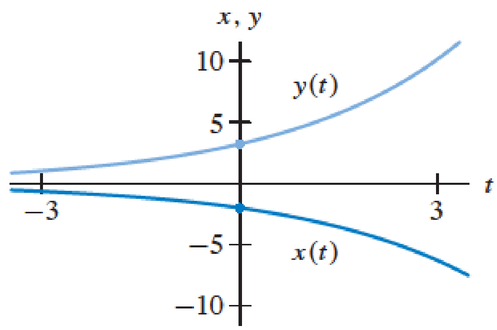
(d) One eigenvector $\mathbf{V}_1$ for the eigenvalue λ_1 is $\mathbf{V}_1 = (2, -1 + \sqrt{5})$, and one eigenvector $\mathbf{V}_2$ for the eigenvalue λ_2 is $\mathbf{V}_2 = (-2, 1 + \sqrt{5})$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{(3+\sqrt{5})t/2} \begin{pmatrix} 2 \\ -1 + \sqrt{5} \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{(3-\sqrt{5})t/2} \begin{pmatrix} -2 \\ 1 + \sqrt{5} \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{(3+\sqrt{5})t/2} \begin{pmatrix} 2 \\ -1 + \sqrt{5} \end{pmatrix} + k_2 e^{(3-\sqrt{5})t/2} \begin{pmatrix} -2 \\ 1 + \sqrt{5} \end{pmatrix}.$$

12. The characteristic polynomial is

$$(3-\lambda)(-2-\lambda)=0,$$

and therefore the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$.

To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = 3$, we solve the system of equations

$$\begin{cases} 3x_1 = 3x_1 \\ x_1 - 2y_1 = 3y_1 \end{cases}$$

and obtain

$$5y_1 = x_1.$$

Therefore, an eigenvector for the eigenvalue $\lambda_1 = 3$ is $\mathbf{V}_1 = (5, 1)$.

Using the same procedure, we obtain the eigenvector $\mathbf{V}_2 = (0, 1)$ for $\lambda_2 = -2$.

The general solution to this linear system is therefore

$$\mathbf{Y}(t) = k_1 e^{3t} \begin{pmatrix} 5 \\ 1 \end{pmatrix} + k_2 e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(a) We have $\mathbf{Y}(0) = (1, 0)$, so we must find k_1 and k_2 so that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{Y}(0) = k_1 \begin{pmatrix} 5 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous system of linear equations

$$\begin{cases} 5k_1 = 1 \\ k_1 + k_2 = 0. \end{cases}$$

Solving these equations, we obtain $k_1 = 1/5$ and $k_2 = -1/5$. Thus, the particular solution is

$$\mathbf{Y}(t) = \tfrac{1}{5} e^{3t} \begin{pmatrix} 5 \\ 1 \end{pmatrix} - \tfrac{1}{5} e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(b) We have $\mathbf{Y}(0) = (0, 1)$. Since this initial condition is an eigenvector associated to the $\lambda = -2$ eigenvalue, we do not need to do any additional calculation. The desired solution to the initial-value problem is

$$\mathbf{Y}(t) = e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(c) We have $\mathbf{Y}(0) = (2, 2)$, so we must find k_1 and k_2 so that

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \mathbf{Y}(0) = k_1\begin{pmatrix} 5 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous system of linear equations

$$\begin{cases} 5k_1 = 2 \\ k_1 + k_2 = 2. \end{cases}$$

Solving these equations, we obtain $k_1 = 2/5$ and $k_2 = 8/5$. Thus, the particular solution is

$$\mathbf{Y}(t) = \tfrac{2}{5}e^{3t}\begin{pmatrix} 5 \\ 1 \end{pmatrix} + \tfrac{8}{5}e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

16. The characteristic polynomial of $\mathbf{A}$ is

$$(a-\lambda)(d-\lambda)=0,$$

and thus the eigenvalues of $\mathbf{A}$ are $\lambda_1 = a$ and $\lambda_2 = d$.

To find the eigenvectors $\mathbf{V}_1 = (x_1, y_1)$ associated to $\lambda_1 = a$, we need to solve the equation

$$\mathbf{A}\mathbf{V}_1 = a\mathbf{V}_1$$

for all possible vectors $\mathbf{V}_1$. Rewritten in terms of components, this equation is equivalent to

$$\begin{cases} ax_1 + by_1 = ax_1 \\ dy_1 = ay_1. \end{cases}$$

Since $a \neq d$, the second equation implies that $y_1 = 0$. If so, then the first equation is satisfied for all x_1. In other words, the eigenvectors $\mathbf{V}_1$ associated to the eigenvalue a are the vectors of the form $(x_1, 0)$.

To find the eigenvectors $\mathbf{V}_2 = (x_2, y_2)$ associated to $\lambda_2 = d$, we need to solve the equation

$$\mathbf{A}\mathbf{V}_2 = d\mathbf{V}_2$$

for all possible vectors $\mathbf{V}_2$. Rewritten in terms of components, this equation is equivalent to

$$\begin{cases} ax_2 + by_2 = dx_2 \\ dy_2 = dy_2. \end{cases}$$

The second equation always holds, so the eigenvectors $\mathbf{V}_2$ are those vectors that satisfy the equation $ax_2 + by_2 = dx_2$, which can be rewritten as

$$by_2 = (d-a)x_2.$$

These vectors form a line through the origin of slope $(d-a)/b$.

17. The characteristic polynomial of $\mathbf{B}$ is

$$\lambda^2 - (a+d)\lambda + ad - b^2.$$

The roots of this polynomial are

$$\begin{aligned} \lambda &= \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-b^2)}}{2} \\ &= \frac{a+d \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4b^2}}{2} \\ &= \frac{a+d \pm \sqrt{(a-d)^2 + 4b^2}}{2}. \end{aligned}$$

Since the discriminant $D = (a-d)^2 + 4b^2$ is always nonnegative, the roots λ are real. Therefore, the matrix $\mathbf{B}$ has real eigenvalues. If $b \neq 0$, then D is positive and hence $\mathbf{B}$ has two distinct eigenvalues. (The only way to have only one eigenvalue is for $D = 0$).

18. The characteristic equation is

$$(a - \lambda)(-\lambda) - bc = \lambda^2 - a\lambda - bc = 0.$$

Finding the roots via the quadratic formula, we obtain the eigenvalues

$$\frac{a \pm \sqrt{a^2 + 4bc}}{2}.$$

Note that these eigenvalues are very different from the case where the matrix is upper triangular (see Exercise 16). For example, they are not necessarily real numbers because $a^2 + 4bc$ can be negative.