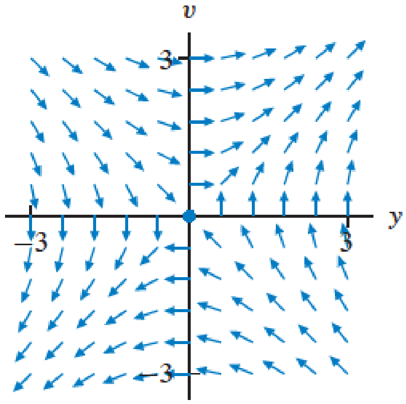
7. **(a)** Let $v = dy/dt$. Then

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = y.$$

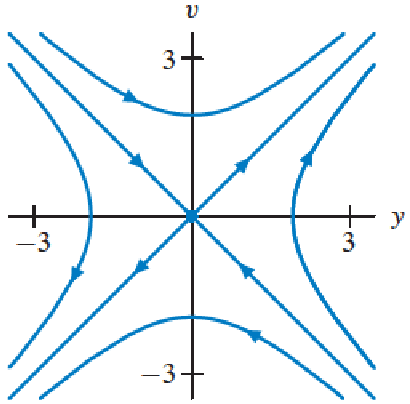
Thus the associated vector field is $\mathbf{V}(y, v) = (v, y)$.

(b) See part (c).

(c)



(d)



(e) As t increases, solutions in the 2nd and 4th quadrants move toward the origin and away from the line $y = -v$. Solutions in the 1st and 3rd quadrants move away from the origin and toward the line $y = v$.

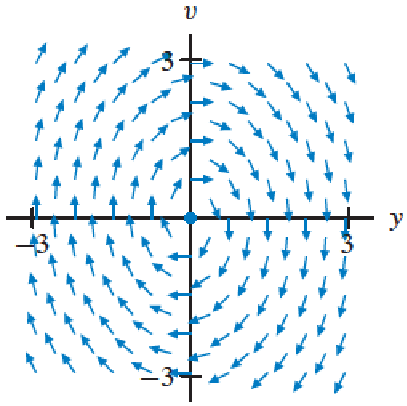
8. **(a)** Let $v = dy/dt$. Then

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = -2y.$$

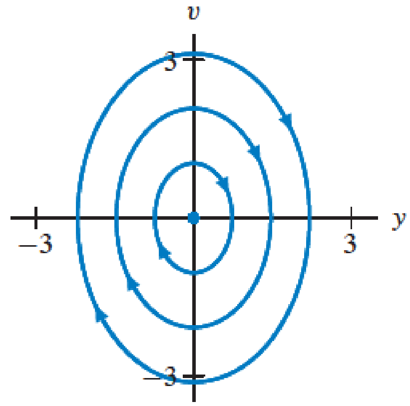
Thus the associated vector field is $\mathbf{V}(y, v) = (v, -2y)$.

(b) See part (c).

(c)



(d)



(e) As t increases, solutions move around the origin on ovals in the clockwise direction.

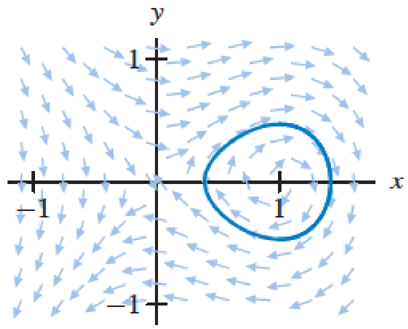
11. **(a)** There are equilibrium points at $(\pm 1, 0)$, so only systems (ii) and (vii) are possible. Since the direction field points toward the x-axis if $y \neq 0$, the equation $dy/dt = y$ does not match this field. Therefore, system (vii) is the system that generated this direction field.

(b) The origin is the only equilibrium point, so the possible systems are (iii), (iv), (v), and (viii). The direction field is not tangent to the y-axis, so it does not match either system (iv) or (v). Vectors point toward the origin on the line $y = x$, so $dy/dt = dx/dt$ if $y = x$. This condition is not satisfied by system (iii). Consequently, this direction field corresponds to system (viii).

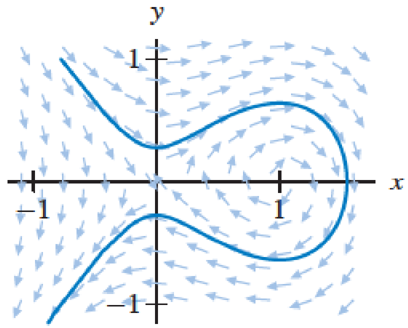
(c) The origin is the only equilibrium point, so the possible systems are (iii), (iv), (v), and (viii). Vectors point directly away from the origin on the y-axis, so this direction field does not correspond to systems (iii) and (viii). Along the line $y = x$, the vectors are more vertical than horizontal. Therefore, this direction field corresponds to system (v) rather than system (iv).

(d) The only equilibrium point is $(1, 0)$, so the direction field must correspond to system (vi).

21. **(a)** The $x(t)$- and $y(t)$-graphs are periodic, so they correspond to a solution curve that returns to its initial condition in the phase plane. In other words, its solution curve is a closed curve. Since the amplitude of the oscillation of $x(t)$ is relatively large, these graphs must correspond to the outermost closed solution curve.

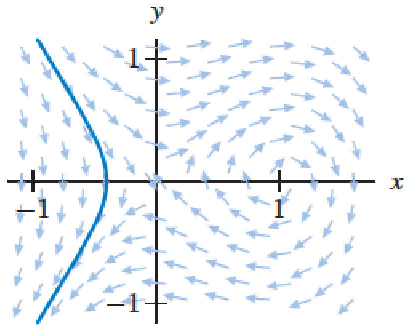


(b) The graphs are not periodic, so they cannot correspond to the two closed solution curves in the phase portrait. Both graphs cross the t-axis. The value of $x(t)$ is initially negative, then becomes positive and reaches a maximum, and finally becomes negative again. Therefore, the corresponding solution curve is the one that starts in the second quadrant, then travels through the first and fourth quadrants, and finally enters the third quadrant.

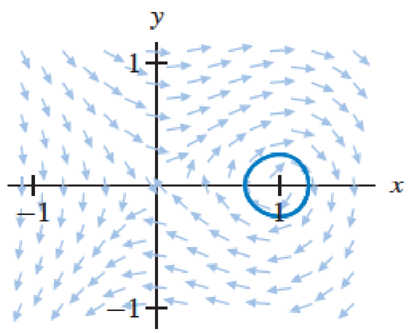


(c) The graphs are not periodic, so they cannot correspond to the two closed solution curves in the phase portrait. Only one graph crosses the t-axis. The other graph remains negative for all time. Note that the two graphs cross.

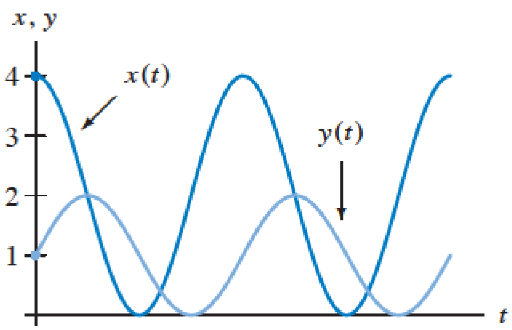
The corresponding solution curve is the one that starts in the second quadrant and crosses the x-axis and the line $y = x$ as it moves through the third quadrant.



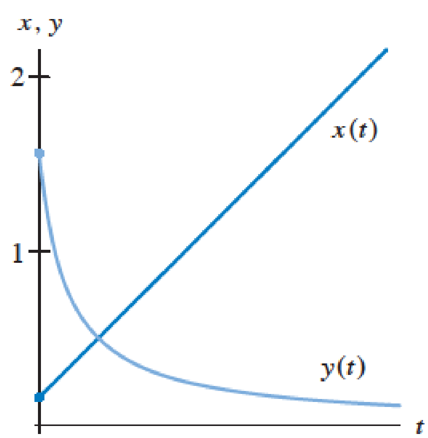
(d) The $x(t)$- and $y(t)$-graphs are periodic, so they correspond to a solution curve that returns to its initial condition in the phase plane. In other words, its solution curve is a closed curve. Since the amplitude of the oscillation of $x(t)$ is relatively small, these graphs must correspond to the intermost closed solution curve.



24. Since the solution curve is an ellipse that is centered at (2, 1), the $x(t)$- and $y(t)$-graphs are periodic. They oscillate about the lines $x = 2$ and $y = 1$.



26. The $x(t)$-graph starts with a small positive value and increases as t increases. The $y(t)$-graph starts at approximately 1.6 and decreases as t increases. However, $y(t)$ remains positive for all t.



2. To check that $dx/dt = 2x + 2y$, we compute both

$$\frac{dx}{dt} = 6e^{2t} + e^t$$

and

$$2x + 2y = 6e^{2t} + 2e^t - 2e^t + 2e^{4t} = 6e^{2t} + 2e^{4t}.$$

Since the results of these two calculations do not agree, the first equation in the system is not satisfied, and $(x(t), y(t))$ is not a solution.

5. The second equation in the system is $dy/dt = -y$, and from Section 1.1, we know that $y(t)$ must be a function of the form y_0e^{-t}, where y_0 is the initial value.

7. From the second equation, we know that $y(t) = k_1 e^{-t}$ for some constant k_1. Using this observation, the first equation in the system can be rewritten as

$$\frac{dx}{dt} = 2x + k_1 e^{-t}.$$

This equation is a first-order linear equation, and we can derive the general solution using the Extended Linearity Principle from Section 1.8 or integrating factors from Section 1.9.

Using the Extended Linearity Principle, we note that the general solution of the associated homogeneous equation is $x_h(t) = k_2 e^{2t}$.

To find one solution to the nonhomogeneous equation, we guess $x_p(t) = \alpha e^{-t}$. Then

$$\begin{aligned} \frac{dx_p}{dt} - 2x_p &= -\alpha e^{-t} - 2\alpha e^{-t} \\ &= -3\alpha e^{-t}. \end{aligned}$$

Therefore, $x_p(t)$ is a solution if $\alpha = -k_1/3$.

The general solution for $x(t)$ is

$$x(t) = k_2 e^{2t} - \frac{k_1}{3} e^{-t}.$$

8. **(a)** No. Given the general solution

$$\left(k_2 e^{2t} - \frac{k_1}{3} e^{-t}, k_1 e^{-t} \right),$$

the function $y(t) = 3e^{-t}$ implies that $k_1 = 3$. But this choice of k_1 implies that the coefficient of e^{-t} in the formula for $x(t)$ is -1 rather than $+1$.

(b) To determine that $\mathbf{Y}(t)$ is not a solution without reference to the general solution, we check the equation $dx/dt = 2x + y$. We compute both

$$\frac{dx}{dt} = -e^{-t}$$

and

$$2x + y = 2e^{-t} + 3e^{-t}.$$

Since these two functions are not equal, $\mathbf{Y}(t)$ is not a solution.

2. **(a)** We compute

$$\frac{dx}{dt} = \frac{d(e^{2t})}{dt} = 2e^{2t} = 2x \quad \text{and} \quad \frac{dy}{dt} = \frac{d(3e^t)}{dt} = 3e^t = y,$$

so $(e^{2t}, 3e^t)$ is a solution.

(b)

Table 2.3

t	Euler's approx.	actual	distance
0	(1, 3)	(1, 3)	
2	(16, 15.1875)	(54.59, 22.17)	39.22
4	(256, 76.88)	(2981, 164)	2726
6	(4096, 389)	(162755, 1210)	158661

(c)

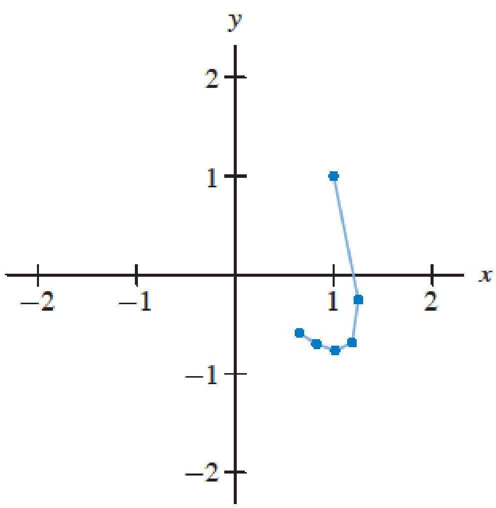
Table 2.4

t	Euler's approx.	actual	distance
0	(1, 3)	(1, 3)	
2	(38.34, 20.18)	(54.59, 22.17)	16.38
4	(1470, 136)	(2981, 164)	1511.4
6	(56347, 913)	(162755, 1210)	106408

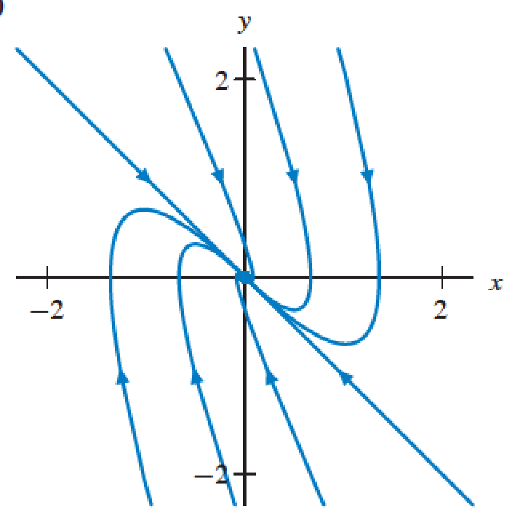
(d) The solution curve starts at (1, 3) and tends to infinity in both the x- and y-directions. Because the solution is an exponential, Euler's method has a hard time keeping up with the growth of the solutions.

3. **(a)** Euler approximation yields $(x_5, y_5) \approx (0.65, -0.59)$.

(b)



(c)



2. Note that $dy/dt > 0$ for all (x, y). Hence, there are no equilibrium points for this system.

3. Let $v = dy/dt$. Then $dv/dt = d^2y/dt^2$, and we obtain the system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = 1.$$

7. First, we check to see if $dx/dt = 2x - 2y^2$ is satisfied. We compute

$$\frac{dx}{dt} = -6e^{-6t} \quad \text{and} \quad 2x - 2y^2 = 2e^{-6t} - 8e^{-6t} = -6e^{-6t}.$$

Second, we check to see if $dy/dt = -3y$. We compute

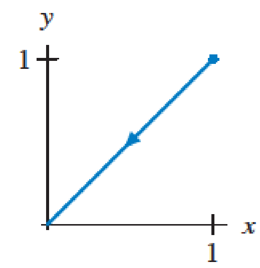
$$\frac{dy}{dt} = -6e^{-3t} \quad \text{and} \quad -3y = -3(2e^{-3t}) = -6e^{-3t}.$$

Since both equations are satisfied, $(x(t), y(t))$ is a solution.

12. One step of Euler's method is

$$\begin{aligned}(2, 1) + \Delta t\,\mathbf{F}(2, 1) &= (2, 1) + 0.5\,(3, 2)\\ &= (3.5, 2).\end{aligned}$$

13. The point $(1, 1)$ is on the line $y = x$. Along this line, the vector field for the system points toward the origin. Therefore, the solution curve consists of the half-line $y = x$ in the first quadrant. Note that the point $(0, 0)$ is not on this curve.



15. True. First, we check the equation for dx/dt. We have

$$\frac{dx}{dt} = \frac{d(e^{-6t})}{dt} = -6e^{-6t},$$

and

$$2x - 2y^2 = 2(e^{-6t}) - 2(2e^{-3t})^2 = 2e^{-6t} - 8e^{-6t} = -6e^{-6t}.$$

Since that equation holds, we check the equation for dy/dt. We have

$$\frac{dy}{dt} = \frac{d(2e^{-3t})}{dt} = -6e^{-3t},$$

and

$$-3y = -3(2e^{-3t}) = -6e^{-3t}.$$

Since the equations for both dx/dt and dy/dt hold, the function $(x(t), y(t)) = (e^{-6t}, 2e^{-3t})$ is a solution of this system.

16. False. A solution to this system must consist of a pair $(x(t), y(t))$ of functions.

20. True. For an autonomous system, the rates of change of solutions depend only on position, not on time. Hence, if a function $(x_1(t), y_1(t))$ satisfies an autonomous system, then the function given by

$$(x_2(t), y_2(t)) = (x_1(t+T), y_1(t+T)),$$

where T is some constant, satisfies the same system.

30. If x_1 is a root of $f(x)$ (that is, $f(x_1) = 0$), then the line $x = x_1$ is invariant. In other words, given an initial condition of the form (x_1, y), the corresponding solution curve remains on the line for all t. Along the line $x = x_1$, $y(t)$ obeys $dy/dt = g(y)$, so the line $x = x_1$ looks like the phase line of the equation $dy/dt = g(y)$.

Similarly, if $g(y_1) = 0$, then the line $y = y_1$ looks like the phase line for $dx/dt = f(x)$ except that it is horizontal rather than vertical.

Combining these two observations, we see that there will be vertical phase lines in the phase portrait for each root of $f(x)$ and horizontal phase lines in the phase portrait for each root of $g(y)$.