4. The general solution to the associated homogeneous equation is $y_h(t) = ke^{2t}$. For a particular solution of the nonhomogeneous equation, we guess $y_p(t) = \alpha \cos 2t + \beta \sin 2t$. Then

$$\begin{aligned}\frac{dy_p}{dt} - 2y_p &= -2\alpha \sin 2t + 2\beta \cos 2t - 2(\alpha \cos 2t + \beta \sin 2t) \\ &= (2\beta - 2\alpha) \cos 2t + (-2\alpha - 2\beta) \sin 2t.\end{aligned}$$

Consequently, we must have

$$(2\beta - 2\alpha) \cos 2t + (-2\alpha - 2\beta) \sin 2t = \sin 2t$$

for $y_p(t)$ to be a solution, that is, we must solve

$$\begin{cases} -2\alpha - 2\beta = 1 \\ -2\alpha + 2\beta = 0. \end{cases}$$

Hence, $\alpha = -1/4$ and $\beta = -1/4$. The general solution of the nonhomogeneous equation is

$$y(t) = ke^{2t} - \tfrac{1}{4} \cos 2t - \tfrac{1}{4} \sin 2t.$$

8. The general solution to the associated homogeneous equation is $y_h(t) = ke^{2t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha e^{-2t}$. Then

$$\begin{aligned}\frac{dy_p}{dt} - 2y_p &= -2\alpha e^{-2t} - 2\alpha e^{-2t} \\ &= -4\alpha e^{-2t}.\end{aligned}$$

Consequently, we must have $-4\alpha = 3$ for $y_p(t)$ to be a solution. Hence, $\alpha = -3/4$, and the general solution to the nonhomogeneous equation is

$$y(t) = ke^{2t} - \tfrac{3}{4} e^{-2t}.$$

Since $y(0) = 10$, we have

$$10 = k - \tfrac{3}{4},$$

so $k = 43/4$. The function

$$y(t) = \tfrac{43}{4} e^{2t} - \tfrac{3}{4} e^{-2t}$$

is the solution of the initial-value problem.

9. The general solution of the associated homogeneous equation is $y_h(t) = ke^{-t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha \cos 2t + \beta \sin 2t$. Then

$$\begin{aligned}\frac{dy_p}{dt} + y_p &= -2\alpha \sin 2t + 2\beta \cos 2t + \alpha \cos 2t + \beta \sin 2t \\ &= (\alpha + 2\beta) \cos 2t + (-2\alpha + \beta) \sin 2t.\end{aligned}$$

Consequently, we must have

$$(\alpha + 2\beta) \cos 2t + (-2\alpha + \beta) \sin 2t = \cos 2t$$

for $y_p(t)$ to be a solution. We must solve

$$\begin{cases} \alpha + 2\beta = 1 \\ -2\alpha + \beta = 0. \end{cases}$$

Hence, $\alpha = 1/5$ and $\beta = 2/5$. The general solution to the differential equation is

$$y(t) = ke^{-t} + \tfrac{1}{5} \cos 2t + \tfrac{2}{5} \sin 2t.$$

To find the solution of the given initial-value problem, we evaluate the general solution at $t = 0$ and obtain

$$y(0) = k + \tfrac{1}{5}.$$

Since the initial condition is $y(0) = 5$, we see that $k = 24/5$. The desired solution is

$$y(t) = \tfrac{24}{5} e^{-t} + \tfrac{1}{5} \cos 2t + \tfrac{2}{5} \sin 2t.$$

17. **(a)** We compute

$$\frac{dy_1}{dt} = \frac{1}{(1-t)^2} = (y_1(t))^2$$

to see that $y_1(t)$ is a solution.

(b) We compute

$$\frac{dy_2}{dt} = 2\frac{1}{(1-t)^2} \neq (y_2(t))^2$$

to see that $y_2(t)$ is not a solution.

(c) The equation $dy/dt = y^2$ is not linear. It contains y^2.

18. **(a)** The constant function $y(t) = 2$ for all t is an equilibrium solution.

(b) If $y(t) = 2 - e^{-t}$, then $dy/dt = e^{-t}$. Also, $-y(t) + 2 = e^{-t}$. Consequently, $y(t) = 2 - e^{-t}$ is a solution.

(c) Note that the solution $y(t) = 2 - e^{-t}$ has initial condition $y(0) = 1$. If the Linearity Principle held for this equation, then we could multiply the equilibrium solution $y(t) = 2$ by $1/2$ and obtain another solution that satisfies the initial condition $y(0) = 1$. Two solutions that satisfy the same initial condition would violate the Uniqueness Theorem.

20. If $y_p(t) = at^2 + bt + c$, then

$$\begin{aligned}\frac{dy_p}{dt} + 2y_p &= 2at + b + 2at^2 + 2bt + 2c \\ &= 2at^2 + (2a + 2b)t + (b + 2c).\end{aligned}$$

Then $y_p(t)$ is a solution if this quadratic is equal to $3t^2 + 2t - 1$. In other words, $y_p(t)$ is a solution if

$$\begin{cases} 2a = 3 \\ 2a + 2b = 2 \\ b + 2c = -1. \end{cases}$$

From the first equation, we have $a = 3/2$. Then from the second equation, we have $b = -1/2$. Finally, from the third equation, we have $c = -1/4$. The function

$$y_p(t) = \tfrac{3}{2}t^2 - \tfrac{1}{2}t - \tfrac{1}{4}$$

is a solution of the differential equation.

31. Step 1: *Before retirement*

First we calculate how much money will be in her retirement fund after 30 years. The differential equation modeling the situation is

$$\frac{dy}{dt} = .07y + 5{,}000,$$

where $y(t)$ represents the fund's balance at time t.

The general solution of the homogeneous equation is $y_h(t) = ke^{0.07t}$.

To find a particular solution, we observe that the nonhomogeneous equation is autonomous and that it has an equilibrium solution at $y = -5{,}000/0.07 \approx -71{,}428.57$. We can use this equilibrium solution as the particular solution. (It is the solution we would have computed if we had guessed a constant solution). We obtain

$$y(t) = ke^{0.07t} - 71{,}428.57.$$

From the initial condition, we see that $k = 71{,}428.57$, and

$$y(t) = 71{,}428.57(e^{0.07t} - 1).$$

Letting $t = 30$, we compute that the fund contains $\approx$ \$511,869.27 after 30 years.

Step 2: *After retirement*

We need a new model for the remaining years since the professor is withdrawing rather than depositing. Since she withdraws at a rate of \$3,000 per month (\$36,000 per year), we write

$$\frac{dy}{dt} = .07y - 36{,}000,$$

where we continue to measure time t in years.

Again, the solution of the homogeneous equation is $y_h(t) = ke^{0.07t}$.

To find a particular solution of the nonhomogeneous equation, we note that the equation is autonomous and that it has an equilibrium at $y = 36{,}000/0.07 \approx 514{,}285.71$. Hence, we may take the particular solution to be this equilibrium solution. (Again, this solution is what we would have computed if we had guessed a constant function for y_p.)

The general solution is

$$y(t) = ke^{0.07t} + 514{,}285.71.$$

In this case, we have the initial condition $y(0) = 511{,}869.27$ since now $y(t)$ is the amount in the fund t years after she retires. Solving $511{,}869.27 = k + 514{,}285.71$, we get $k = -2{,}416.44$. The solution in this case is

$$y(t) = -2{,}416.44e^{0.07t} + 514{,}285.71.$$

Finally, we wish to know when her money runs out. That is, at what time t is $y(t) = 0$? Solving

$$y(t) = -2{,}416.44e^{0.07t} + 514{,}285.71 = 0$$

yields $t \approx 76.58$ years (approximately 919 months).

4. We rewrite the equation in the form

$$\frac{dy}{dt} + 2ty = 4e^{-t^2}$$

and note that the integrating factor is

$$\mu(t) = e^{\int 2t\,dt} = e^{t^2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$e^{t^2}\frac{dy}{dt} + 2te^{t^2}y = 4.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(e^{t^2}y)}{dt} = 4,$$

and integrating both sides with respect to t, we obtain

$$e^{t^2}y = 4t + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = 4te^{-t^2} + ce^{-t^2}.$$

5. Note that the integrating factor is

$$\mu(t) = e^{\int(-2t/(1+t^2))\,dt} = e^{-\ln(1+t^2)} = \left(e^{\ln(1+t^2)}\right)^{-1} = \frac{1}{1+t^2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$\frac{1}{1+t^2}\frac{dy}{dt} - \frac{2t}{(1+t^2)^2}y = \frac{3}{1+t^2}.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d}{dt}\left(\frac{y}{1+t^2}\right) = \frac{3}{1+t^2}.$$

Integrating both sides with respect to t, we obtain

$$\frac{y}{1+t^2} = 3\arctan(t) + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = (1+t^2)(3\arctan(t) + c).$$

9. In Exercise 1, we derived the general solution

$$y(t) = t + \frac{c}{t}.$$

To find the solution that satisfies the initial condition $y(1) = 3$, we evaluate the general solution at $t = 1$ and obtain $c = 2$. The desired solution is

$$y(t) = t + \frac{2}{t}.$$

12. Note that the integrating factor is

$$\mu(t) = e^{\int(-3/t)\,dt} = e^{-3\ln t} = e^{\ln(t^{-3})} = t^{-3}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$t^{-3}\frac{dy}{dt} - 3t^{-4}y = 2e^{2t}.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(t^{-3}y)}{dt} = 2e^{2t},$$

and integrating both sides with respect to t, we obtain

$$t^{-3}y = e^{2t} + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = t^3(e^{2t} + c).$$

To find the solution that satisfies the initial condition $y(1) = 0$, we evaluate the general solution at $t = 1$ and obtain $c = -e^2$. The desired solution is

$$y(t) = t^3(e^{2t} - e^2).$$

19. We rewrite the equation in the form

$$\frac{dy}{dt} - aty = 4e^{-t^2}$$

and note that the integrating factor is

$$\mu(t) = e^{\int(-at)\,dt} = e^{-at^2/2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$e^{-at^2/2}\frac{dy}{dt} - ate^{-at^2/2}y = 4e^{-t^2}e^{-at^2/2}.$$

Applying the Product Rule to the left-hand side and simplifying the right-hand side, we see that this equation is the same as

$$\frac{d(e^{-at^2/2}y)}{dt} = 4e^{-(1+a/2)t^2}.$$

Integrating both sides with respect to t, we obtain

$$e^{-at^2/2}y = \int 4e^{-(1+a/2)t^2}\,dt.$$

The integral on the right-hand side can be expressed in terms of elementary functions only if $1 + a/2 = 0$ (that is, if the factor involving e^{t^2} really isn't there). Hence, the only value of a that yields an integral we can express in terms of elementary functions form is $a = -2$ (see Exercise 4).

22. **(a)** Note that

$$\frac{d\mu}{dt} = \mu(t)(-a(t))$$

by the Fundamental Theorem of Calculus. Therefore, if we rewrite the differential equation as

$$\frac{dy}{dt} - a(t)y = b(t)$$

and multiply the left-hand side of this equation by $\mu(t)$, the left-hand side becomes

$$\begin{aligned}\mu(t)\frac{dy}{dt} - \mu(t)a(t)y &= \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y \\ &= \frac{d(\mu\, y)}{dt}.\end{aligned}$$

Consequently, the function $\mu(t)$ satisfies the requirements of an integrating factor.

(b) To see that $1/\mu(t)$ is a solution of the associated homogeneous equation, we calculate

$$\begin{aligned}\frac{d\left(\frac{1}{\mu(t)}\right)}{dt} &= \frac{-1}{\mu(t)^2}\frac{d\mu}{dt} \\ &= \frac{-1}{\mu(t)^2}\mu(t)(-a(t)) \\ &= a(t)\frac{1}{\mu(t)}.\end{aligned}$$

Thus, $y(t) = 1/\mu(t)$ satisfies the equation $dy/dt = a(t)y$.

(c) To see that $y_p(t)$ is a solution to the nonhomogeneous equation, we compute

$$\begin{aligned}\frac{dy_p}{dt} &= \frac{d\left(\frac{1}{\mu(t)}\right)}{dt}\left(\int_0^t \mu(\tau)\,b(\tau)\,d\tau\right) + \frac{1}{\mu(t)}\mu(t)b(t) \\ &= a(t)\frac{1}{\mu(t)}\left(\int_0^t \mu(\tau)\,b(\tau)\,d\tau\right) + b(t) \\ &= a(t)y_p(t) + b(t).\end{aligned}$$

(d) Let k be an arbitrary constant. Since $k/\mu(t)$ is the general solution of the associated homogeneous equation and

$$\frac{1}{\mu(t)}\int_0^t \mu(\tau)\,b(\tau)\,d\tau$$

is a solution to the nonhomogeneous equation, the general solution of the nonhomogeneous equation is

$$y(t) = \frac{k}{\mu(t)} + \frac{1}{\mu(t)} \int_0^t \mu(\tau)\, b(\tau)\, d\tau$$
$$= \frac{1}{\mu(t)} \left(k + \int_0^t \mu(\tau)\, b(\tau)\, d\tau \right).$$

(e) Since

$$\int \mu(t)\, b(t)\, dt = \int_0^t \mu(\tau)\, b(\tau)\, d\tau + k$$

by the Fundamental Theorem of Calculus, the two formulas agree.

(f) In this equation, $a(t) = -2t$ and $b(t) = 4e^{-t^2}$. Therefore,

$$\mu(t) = e^{\int_0^t 2\tau\, d\tau} = e^{t^2}.$$

Consequently, $1/\mu(t) = e^{-t^2}$. Note that,

$$\frac{d\left(\frac{1}{\mu(t)}\right)}{dt} = (-2t)e^{-t^2} = a(t)\frac{1}{\mu(t)}.$$

Also,

$$y_p(t) = e^{-t^2} \int_0^t e^{\tau^2} \left(4e^{-\tau^2}\right) d\tau = e^{-t^2} \int_0^t 4\, d\tau = 4te^{-t^2}.$$

It is easy to see that $4te^{-t^2}$ satisfies the nonhomogeneous equation.

Therefore, the general solution to the nonhomogeneous equation is

$$ke^{-t^2} + 4te^{-t^2},$$

which can also be written as $(4t + k)e^{-t^2}$. Finally, note that

$$\frac{1}{\mu(t)} \int \mu(t)\, b(t)\, dt = e^{-t^2} \int e^{t^2} \left(4e^{-t^2}\right) dt = (4t + k)e^{-t^2}.$$

3. There are no values of y for which dy/dt is zero for all t. Hence, there are no equilibrium solutions.

4. Since the question only asks for one solution, look for the simplest first. Note that $y(t) = 0$ for all t is an equilibrium solution. There are other equilibrium solutions as well.

10. For $a > -4$, all solutions increase at a constant rate, and for $a < -4$, all solutions decrease at a constant rate. Consequently, a bifurcation occurs at $a = -4$, and all solutions are equilibria.

11. True. We have $dy/dt = e^{-t}$, which agrees with $|y(t)|$.

12. False. A separable equation has the form $dy/dt = g(t)h(y)$. So if $g(t)$ is not constant, then the equation is not separable. For example, $dy/dt = ty$ is separable but not autonomous.

13. True. Autonomous equations have the form $dy/dt = f(y)$. Therefore, we can separate variables by dividing by $f(y)$. That is,

$$\frac{1}{f(y)}\frac{dy}{dt} = 1.$$

14. False. For example, $dy/dt = y + t$ is linear but not separable.

26. **(a)** This equation is linear and nonhomogeneous.

(b) We rewrite the equation in the form

$$\frac{dy}{dt} - \frac{2y}{1+t} = t$$

and note that the integrating factor is

$$\mu(t) = e^{\int -2/(1+t)\,dt} = e^{-2\ln(1+t)} = \frac{1}{(1+t)^2}.$$

Multiplying both sides of the differential equation by $\mu(t)$, we obtain

$$\frac{1}{(1+t)^2}\frac{dy}{dt} - \frac{2y}{(1+t)^3} = \frac{t}{(1+t)^2}.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d}{dt}\left(\frac{y}{(1+t)^2}\right) = \frac{t}{(1+t)^2}.$$

Integrating both sides with respect to t and using the substitution $u = 1 + t$ on the right-hand side, we obtain

$$\frac{y}{(1+t)^2} = \frac{1}{1+t} + \ln|1+t| + k,$$

where k can be any real number. The general solution is

$$y(t) = (1+t) + (1+t)^2\ln|1+t| + k(1+t)^2.$$

49. **(a)** Note that the slopes are constant along vertical lines—lines along which t is constant, so the right-hand side of the corresponding equation depends only on t. The only choices are equations (i) and (iv). Because the slopes are negative for $t > 1$ and positive for $t < 1$, this slope field corresponds to equation (iv).

(b) This slope field has an equilibrium solution corresponding to the line $y = 1$, as does equations (ii), (v), (vii), and (viii). Equations (ii), (v), and (viii) are autonomous, and this slope field is not constant along horizontal lines. Consequently, it corresponds to equation (vii).

(c) This slope field is constant along horizontal lines, so it corresponds to an autonomous equation. The autonomous equations are (ii), (v), and (viii). This field does not correspond to equation (v) because it has the equilibrium solution $y = -1$. The slopes are negative between $y = -1$ and $y = 1$. Consequently, this field corresponds to equation (viii).

(d) This slope field depends both on y and on t, so it can only correspond to equations (iii), (vi), or (vii). It does not correspond to (vii) because it does not have an equilibrium solution at $y = 1$. Also, the slopes are positive if $y > 0$. Therefore, it must correspond to equation (vi).

52. **(a)** The equation is separable. Separating variables and integrating, we obtain

$$\int y^{-2}\,dy = \int -2t\,dt$$
$$-y^{-1} = -t^2 + c,$$

where c is a constant of integration. Multiplying both sides by -1 and inverting yields

$$y(t) = \frac{1}{t^2 + k},$$

where k can be any constant. In addition, the equilibrium solution $y(t) = 0$ for all t is a solution.

(b) If $y(-1) = y_0$, we have

$$y_0 = y(-1) = \frac{1}{1+k}$$

so

$$k = \frac{1}{y_0} - 1.$$

As long as $k > 0$, the denominator is positive for all t, and the solution is bounded for all t. Hence, for $0 \le y_0 < 1$, the solution is bounded for all t. (Note that $y_0 = 0$ corresponds to the equilibrium solution.) All other solutions escape to $\pm\infty$ in finite time.

1. In the case where it takes many predators to eat one prey, the constant in the negative effect term of predators on the prey is small. Therefore, (ii) corresponds the system of large prey and small predators. On the other hand, one predator eats many prey for the system of large predators and small prey, and, therefore, the coefficient of negative effect term on predator-prey interaction on the prey is large. Hence, (i) corresponds to the system of small prey and large predators.