1. **(a)** The function $g_a(t) = 1$ precisely when $u_a(t) = 0$, and $g_a(t) = 0$ precisely when $u_a(t) = 1$, so

$$g_a(t) = 1 - u_a(t).$$

(b) We can compute the Laplace transform of $g_a(t)$ from the definition

$$\mathcal{L}[g_a] = \int_0^a 1e^{-st}\,dt = -\frac{e^{-as}}{s} + \frac{e^{-0s}}{s} = \frac{1}{s} - \frac{e^{-as}}{s}.$$

Alternately, we can use the table

$$\mathcal{L}[g_a] = \mathcal{L}[1 - u_a(t)] = \frac{1}{s} - \frac{e^{-as}}{s}.$$

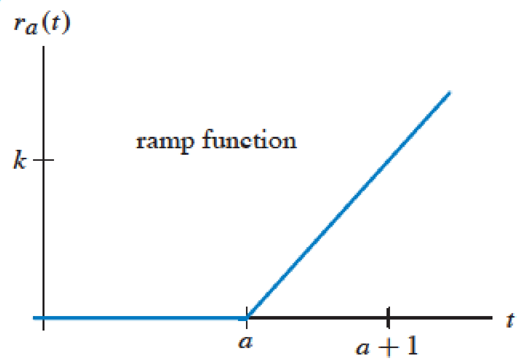
2. **(a)** We have $r_a(t) = u_a(t)y(t-a)$, where $y(t) = kt$. Now

$$\mathcal{L}[y(t)] = k\mathcal{L}[t] = \frac{k}{s^2},$$

so using the rules of Laplace transform,

$$\mathcal{L}[r_a(t)] = \mathcal{L}[u_a(t)y(t-a)] = \frac{k}{s^2}e^{-as}.$$

(b)



4. We have

$$\mathcal{L}[e^{3t}] = \frac{1}{s-3},$$

so using the rule

$$\mathcal{L}[u_a(t)y(t-a)] = e^{-as}\mathcal{L}[y(t)],$$

we determine that

$$\mathcal{L}[u_2(t)e^{3(t-2)}] = \frac{e^{-2s}}{s-3}.$$

The desired function is $u_2(t)e^{3(t-2)}$.

8. Taking the Laplace transform of both sides of the equation gives us

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \mathcal{L}[u_2(t)],$$

so

$$s\mathcal{L}[y] - y(0) = \frac{e^{-2s}}{s}.$$

Substituting the initial condition yields

$$s\mathcal{L}[y] - 3 = \frac{e^{-2s}}{s},$$

so that

$$\mathcal{L}[y] = \frac{e^{-2s}}{s^2} + \frac{3}{s}.$$

By taking the inverse of the Laplace transform, we get

$$y(t) = u_2(t)(t-2) + 3.$$

To check our answer, we compute

$$\frac{dy}{dt} = \frac{du_2}{dt}(t-2) + u_2(t),$$

and since $du_2/dt = 0$ except at $t = 2$ (where it is undefined),

$$\frac{dy}{dt} = u_2(t).$$

Hence, our $y(t)$ satisfies the differential equation except when $t = 2$. (We cannot expect $y(t)$ to satisfy the differential equation at $t = 2$ because the differential equation is not continuous there.) Note that $y(t)$ also satisfies the initial condition $y(0) = 3$.

15. Taking the Laplace transform of both sides of the equation, we have

$$\mathcal{L}\left[\frac{dy}{dt}\right] = -\mathcal{L}[y] + \mathcal{L}[u_a(t)],$$

which is equivalent to

$$s\mathcal{L}[y] - y(0) = -\mathcal{L}[y] + \frac{e^{-as}}{s}.$$

Solving for $\mathcal{L}[y]$ yields

$$\mathcal{L}[y] = \frac{e^{-as}}{s(s+1)} + \frac{y(0)}{s+1}.$$

Using the partial fractions decomposition

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1},$$

we get

$$\mathcal{L}[y] = \frac{e^{-as}}{s} - \frac{e^{-as}}{s+1} + \frac{y(0)}{s+1}.$$

Taking the inverse Laplace transform, we obtain

$$\begin{aligned} y(t) &= u_a(t) - u_a(t)e^{-(t-a)} + y(0)e^{-t} \\ &= u_a(t)\left(1 - e^{-(t-a)}\right) + y(0)e^{-t}. \end{aligned}$$

To check our answer, we compute

$$\frac{dy}{dt} = \frac{du_a}{dt}\left(1 - e^{-(t-a)}\right) + u_a(t)e^{-(t-a)} - y(0)e^{-t}$$

and since $du_a/dt = 0$ except at $t = a$ (where it is undefined),

$$\begin{aligned} \frac{dy}{dt} + y &= u_a(t)e^{-(t-a)} - y(0)e^{-t} + u_a(t)\left(1 - e^{-(t-a)}\right) + y(0)e^{-t} \\ &= u_a(t). \end{aligned}$$

Hence, our $y(t)$ satisfies the differential equation except when $t = a$. (We cannot expect $y(t)$ to satisfy the differential equation at $t = a$ because the differential equation is not continuous there.)

16. We can write $\mathcal{L}[f]$ as the sum of two integrals, that is,

$$\mathcal{L}[f] = \int_0^\infty f(t)\,e^{-st}\,dt = \int_0^T f(t)\,e^{-st}\,dt + \int_T^\infty f(t)\,e^{-st}\,dt.$$

Next, we use the substitution $u = t - T$ on the second integral. Note that $t = u + T$. We get

$$\int_T^\infty f(t)\,e^{-st}\,dt = \int_0^\infty f(u+T)\,e^{-s(u+T)}\,du.$$

Since f is periodic with period T, we can rewrite the last integral as

$$e^{-Ts}\int_0^\infty f(u)\,e^{-su}\,du,$$

which is just $e^{-Ts}\mathcal{L}[f]$. Hence,

$$\mathcal{L}[f] = \int_0^T f(t)\,e^{-st}\,dt + e^{-Ts}\mathcal{L}[f].$$

We have

$$(1 - e^{-Ts})\mathcal{L}[f] = \int_0^T f(t)\,e^{-st}\,dt.$$

Consequently,

$$\mathcal{L}[f] = \frac{1}{1 - e^{-Ts}}\int_0^T f(t)\,e^{-st}\,dt.$$

18. From the formula in Exercise 16, we see that we need only compute the integral $\int_0^1 te^{-st}\,dt$. Using integration by parts (as in Exercise 2 of Section 6.1), we get

$$\mathcal{L}[z] = \frac{1}{1 - e^{-s}}\left(\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}\right)$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s(1 - e^{-s})}.$$

5. Using the formula

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s^2\mathcal{L}[y] - y'(0) - sy(0),$$

and the linearity of the Laplace transform, we get that

$$s^2\mathcal{L}[y] - y'(0) - sy(0) + \omega^2\mathcal{L}[y] = 0.$$

Substituting the initial conditions and solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{s}{s^2 + \omega^2}.$$