2. **(a)** If $H(x, y) = \sin(xy)$, then

$$\frac{\partial H}{\partial x} = y\cos(xy)$$

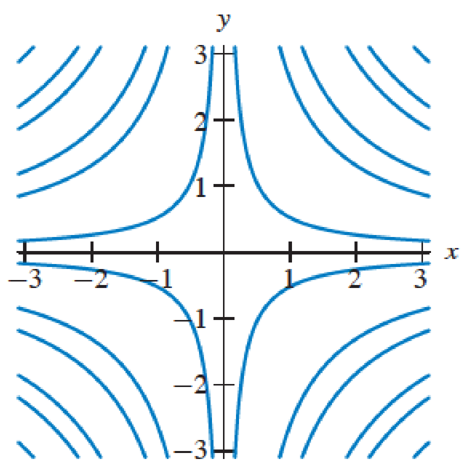
and so

$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}.$$

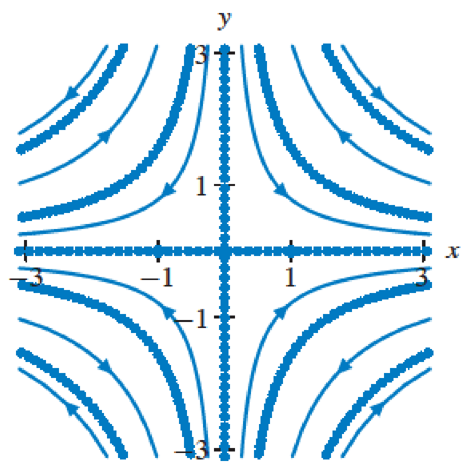
Similarly,

$$\frac{\partial H}{\partial y} = x\cos(xy) = \frac{dx}{dt}.$$

(b) Note that the level sets of H are the same curves as those of the level sets of xy.



(c) Note that there are many curves of equilibrium points for this system: besides the origin, whenever $xy = n\pi + \pi/2$, the vector field vanishes.



9. We know that the equilibrium points of a Hamiltonian system cannot be sources or sinks. Phase portrait (b) has a spiral source, so it is not Hamiltonian. Phase portrait (c) has a sink and a source, so it is not Hamiltonian. Phase portraits (a) and (d) might come from Hamiltonian systems. (Try to imagine a function which has the solution curves as level sets.)

12. First we check to see if the partial derivative with respect to x of the first component of the vector field is the negative of the partial derivative with respect to y of the second component. We have

$$\frac{\partial 1}{\partial x} = 0$$

while

$$-\frac{\partial y}{\partial y} = -1.$$

Since these are not equal, the system is not Hamiltonian.

13. First we check to see if the partial derivative with respect to x of the first component of the vector field is the negative of the partial derivative with respect to y of the second component. We have

$$\frac{\partial (x \cos y)}{\partial x} = \cos y$$

while

$$-\frac{\partial (-y \cos x)}{\partial y} = \cos x.$$

Since these two are not equal, the system is not Hamiltonian.

14. First note that

$$\frac{\partial F(y)}{\partial x} = 0 = -\frac{\partial G(x)}{\partial y},$$

that is, the partial derivative of the x component of the vector field with respect to x is equal to the negative of the partial derivative of the y component with respect to y. Hence, the system is Hamiltonian. Integrating the x component of the vector field with respect to y yields

$$H(x, y) = \int F(y)\, dy + c$$

where the "constant" c could depend on x. If we differentiate this H with respect to x we get

$$-\frac{\partial H}{\partial x} = -c'(x).$$

Thus we take $c = -\int G(x)\, dx$. A Hamiltonian function is

$$H(x, y) = \int F(y)\, dy - \int G(x)\, dx.$$

18. **(a)** We have

$$\frac{\partial H}{\partial y} = y \quad \text{and} \quad -\frac{\partial H}{\partial x} = x^2 - a,$$

so this system is Hamiltonian with the given function H.

(b) Note that $dx/dt = 0$ if and only if $y = 0$ and $dy/dt = 0$ if and only if $x = \pm\sqrt{a}$. Consequently if $a < 0$, then there are no equilibrium points. If $a = 0$, there is one equilibrium point at $(0, 0)$ and if $a > 0$, there are two equilibrium points at $(\pm\sqrt{a}, 0)$.

(c) The Jacobian matrix is

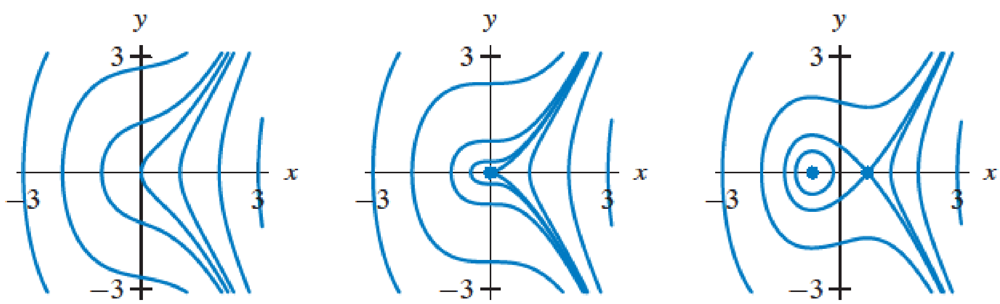
$$\begin{pmatrix} 0 & 1 \\ 2x & 0 \end{pmatrix},$$

which, when evaluated at the equilibrium points, becomes

$$\begin{pmatrix} 0 & 1 \\ \pm 2\sqrt{a} & 0 \end{pmatrix}.$$

At $(\sqrt{a}, 0)$, the eigenvalues are $\pm\sqrt{2\sqrt{a}}$ so this equilibrium point is a saddle. At $(-\sqrt{a}, 0)$, the eigenvalues are $\pm i\sqrt{2\sqrt{a}}$ so this equilibrium point is a center. If $a = 0$ the eigenvalues are both 0, so this point is a node.

(d)



Phase portrait for $a < 0$ Phase portrait for $a = 0$ Phase portrait for $a > 0$

(e) As a increases toward 0, the phase portrait changes from having no equilibrium points to having a single equilibrium point at $a = 0$. If $a > 0$, there is a pair of equilibrium points.

1. Since the equilibrium point is at the origin and the system has only polynomial terms, the linearized system is just the linear terms in dx/dt and dy/dt, that is,

$$\frac{dx}{dt} = x$$
$$\frac{dy}{dt} = -2y.$$

2. From the linearized system in Exercise 1, we see (without any calculation) that the eigenvalues are 1 and -2. Hence, the origin is a saddle.

6. This system is not a Hamiltonian system. If it were, then we would have

$$\frac{\partial H}{\partial y} = \frac{dx}{dt} \quad \text{and} \quad -\frac{\partial H}{\partial x} = \frac{dy}{dt}$$

for some function $H(x, y)$. In that case, equality of mixed partials would imply that

$$\frac{\partial}{\partial x}\left(\frac{dx}{dt}\right) = -\frac{\partial}{\partial y}\left(\frac{dy}{dt}\right).$$

For this system, we have

$$\frac{\partial}{\partial x}\left(\frac{dx}{dt}\right) = 2y \quad \text{and} \quad -\frac{\partial}{\partial y}\left(\frac{dy}{dt}\right) = -2y.$$

Since these two partials do not agree, no such function $H(x, y)$ exists.

7. This system is not a gradient system. If it were, then we would have

$$\frac{\partial G}{\partial x} = \frac{dx}{dt} \quad \text{and} \quad \frac{\partial G}{\partial y} = \frac{dy}{dt}$$

for some function $G(x, y)$. In that case, equality of mixed partials would imply that

$$\frac{\partial}{\partial y}\left(\frac{dx}{dt}\right) = \frac{\partial}{\partial x}\left(\frac{dy}{dt}\right).$$

For this system, we have

$$\frac{\partial}{\partial y}\left(\frac{dx}{dt}\right) = 2x + 2y \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{dy}{dt}\right) = 2x.$$

Since these two partials do not agree, no such function $G(x, y)$ exists.

8. Some possibilities are:

- The solution is unbounded. That is, either $|x(t)| \to \infty$ or $|y(t)| \to \infty$ (or both) as t increases.
- Similarly, $x(t)$ or $y(t)$ (or both) oscillate with increasing amplitude as t increases (similar to $t \sin t$).
- The solution tends to an equilibrium point.
- The solution tends to a periodic solution, as in the Van der Pol equation (see Section 5.1).
- The solution tends to a curve consisting of equilibrium points and solutions connecting equilibrium points.

9. If the system is a linear system, then all nonequilibrium solutions tend to infinity as t increases, that is, $|\mathbf{Y}(t)| \to \infty$ as $t \to \infty$.

If the system is not linear, it is possible for a solution to spiral toward a periodic solution. For example, consider the Van der Pol equation discussed in Section 5.1. (These two behaviors are the only possibilities.)

11. True. The x-nullcline is where $dx/dt = 0$ and the y-nullcline is where $dy/dt = 0$, so any point in common must be an equilibrium point.

12. False. For example, both nullclines for the system

$$\frac{dx}{dt} = x - y$$
$$\frac{dy}{dt} = y - x$$

are the line $y = x$. Moreover, since the nullclines are identical, all points on the line are equilibrium points.

26. **(a)** Letting $y = dx/dt$, we obtain the system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = 3x - x^3 - 2y.$$

From the first equation, we see that $y = 0$ for any equilibrium point. Substituting $y = 0$ in the equation $3x - x^3 - 2y = 0$ yields $x = 0$ or $x^2 = 3$. Hence, the equilibria are $(0, 0)$ and $(\pm\sqrt{3}, 0)$.

(b) The Jacobian matrix is

$$\begin{pmatrix} 0 & 1 \\ 3 - 3x^2 & -2 \end{pmatrix}.$$

Evaluating the Jacobian at $(0, 0)$ yields

$$\begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix},$$

which has eigenvalues -3 and 1. Hence, the origin is a saddle. At $(\pm\sqrt{3}, 0)$, the Jacobian matrix is

$$\begin{pmatrix} 0 & 1 \\ -6 & -2 \end{pmatrix},$$

which has eigenvalues $-1 \pm i\sqrt{5}$. Hence, these two equilibria are spiral sinks.

27. To see if the system is Hamiltonian, we compute

$$\frac{\partial(-3x+10y)}{\partial x} = -3 \quad \text{and} \quad -\frac{\partial(-x+3y)}{\partial y} = -3.$$

Since these partials agree, the system is Hamiltonian.

To find the Hamiltonian function, we use the fact that

$$\frac{\partial H}{\partial y} = \frac{dx}{dt} = -3x + 10y.$$

Integrating with respect to y gives

$$H(x, y) = -3xy + 5y^2 + \phi(x),$$

where $\phi(x)$ represents the terms whose derivative with respect to y are zero. Differentiating this expression for $H(x, y)$ with respect to x gives

$$-3y + \phi'(x) = -\frac{dy}{dt} = x - 3y.$$

We choose $\phi(x) = \frac{1}{2}x^2$ and obtain the Hamiltonian function

$$H(x, y) = -3xy + 5y^2 + \frac{x^2}{2}.$$

We know that the solution curves of a Hamiltonian system remain on the level sets of the Hamiltonian function. Hence, solutions of this system satisfy the equation

$$-3xy + 5y^2 + \frac{x^2}{2} = h$$

for some constant h. Multiplying through by 2 yields the equation

$$x^2 - 6xy + 10y^2 = k$$

where $k = 2h$ is a constant.