Math 312 Spring 2016 **SHOW ALL YOUR WORK & EXPLAIN EVERY ANSWER** Quiz SEVEN

1. (*10 points*) One interesting application of contour integration is the ability to find the area of odd shapes in the plane. If we denote the area enclosed by a positively-oriented contour C by A, then

$$A = \frac{1}{2i}\oint_C \overline{z}\,dz$$

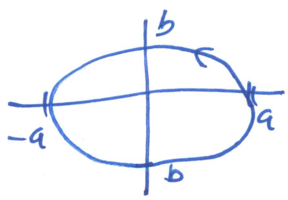
(a) (*6 points*) Recalling that the parametrization given by $z(t) = a\cos t + ib\sin t, 0 \le t \le 2\pi$ represents an elliptical contour C with horizontal axis a and vertical axis b use the formula for A to compute the area enclosed by an ellipse. (Your final answer should only involve π, a and b.)

$$A = \frac{1}{2i}\int_0^{2\pi} (a\cos t - ib\sin t)(-a\sin t + ib\cos t)\,dt$$

$$A = \frac{1}{2i}\int_0^{2\pi} a^2\cos t\sin t + b^2\sin t\cos t + i(ab\sin^2 t + ab\cos^2 t)\,dt$$

$z = a\cos t + ib\sin t, 0 \le t \le 2\pi$

$z' = -a\sin t + ib\cos t$



$$A = \frac{1}{2i}(a^2+b^2)\int_0^{2\pi} \cos t\sin t\,dt + \frac{1}{2i}ab\int_0^{2\pi} i\,dt$$

$$A = \frac{a^2+b^2}{2i}\int_0^{2\pi} \frac{\sin 2t}{2}\,dt + \frac{1}{2i}ab\cdot 2\pi i$$

$$= \frac{a^2+b^2}{4i}\left(-\frac{\cos 2t}{2}\right)\Big|_0^{2\pi} + \pi ab$$

$$= \frac{a^2+b^2}{8i}(\cos 0 - \cos 2\pi) + \pi ab$$

$$= \pi ab$$

(b) (*4 points*) On the same contour as part **(a)** find the value of B, where

$$B = \frac{1}{2i}\oint_C z\,dz$$

(HINT: think how the functions in the integrands of A and B are different to obtain the value B of this integral without doing very much work.) EXPLAIN YOUR ANSWER.

z is analytic everywhere so by the Cauchy-Goursat Theorem $B = \frac{1}{2i}\oint_C z\,dz = 0$ regardless of the shape of C

We know z is analytic because it's a polynomial in z and all polynomials in z are ENTIRE (analytic everywhere)