Math 312 Spring 2016 **SHOW ALL YOUR WORK & EXPLAIN EVERY ANSWER** Quiz NINE

Fall 2001 Final Exam, Question 9. Evaluate $I = \int_0^{2\pi} \sin^4\theta \, d\theta$ using contour integration.

(a) *[4 points]* Show that I can be written as the contour integral $J = \frac{1}{16i} \oint_{|z|=1} \frac{(z^2-1)^4}{z^5} dz$.

[HINT: Use the fact that $z = e^{i\theta}$ to transform I into J.]

Let $z = e^{i\theta}$, $|z| = 1$

$\sin\theta = \left(z - \frac{1}{z}\right)\frac{1}{2i}$ $\quad \sin^4\theta = \left[\left(z - \frac{1}{z}\right)\frac{1}{2i}\right]^4$

$= \left(\frac{z^2-1}{z}\right)^4 \frac{1}{2^4 \cdot i^4} = \frac{(z^2-1)^4}{z^4}\frac{1}{16}$

$I = \int_0^{2\pi} \sin^4\theta \, d\theta = \oint_{|z|=1} \frac{(z^2-1)^4}{z^4} \frac{1}{16} \frac{dz}{iz}$

$z = e^{i\theta}$ $\quad dz = ie^{i\theta}d\theta = iz\,d\theta$

$= \frac{1}{16i} \oint \frac{(z^2-1)^4}{z^5} dz$

(b) *[4 points]* Show that the residue of the integrand $f(z) = \frac{(z^2-1)^4}{z^5}$ at $z = 0$ is 6. **What is the order of this singularity at** $z = 0$? [NOTE: $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.]

$f(z) = \frac{(z^2-1)^4}{z^5} = \frac{(z^2)^4 + 4(z^2)^3(-1) + 6(z^2)^2(-1)^2 + 4z^2(-1)^3 + (-1)^4}{z^5}$

$f(z) = \frac{z^8 - 4z^6 + 6z^4 - 4z^2 + 1}{z^5} = z^3 - 4z + \frac{6}{z} - \frac{4}{z^3} + \frac{1}{z^5}$

Thus $f(z)$ has a pole of order 5 at $z=0$
and has a residue of 6 (coefficient of $\frac{1}{z}$ term)

[You could also use $\lim_{z\to 0} \frac{1}{4!} \frac{d^4}{dz^4}\left(\frac{(z^2-1)^4 z^5}{z^5}\right) = 6$]

(c) *[2 points]* Compute the value of I by using Cauchy's Residue Theorem to evaluate J. (HINT: How are I and J related to each other?)

$I = J = \frac{1}{16i} 2\pi i \, \mathrm{Res}(f, 0)$

$= \frac{\pi}{8} \cdot 6 = \frac{3\pi}{4}$