Math 312 Spring 2016 **SHOW ALL YOUR WORK** Quiz **FOUR**

1. (*10 points*) We want to find a formula for an **entire** function $f(z)$ but all we know is that its real part is given by $u(x,y) = x^3 - 3xy^2 - 4xy + 6$ and that it maps the point $(1,1)$ to the origin.

(a) (*6 points*) Use the Cauchy-Riemann Equations (i.e. $u_x = v_y$, $u_y = -v_x$) to find the imaginary part of $f(z)$, sometimes written as $v(x,y)$, *exactly.*

$u_x = 3x^2 - 3y^2 - 4y = v_y$ (CRE)

reverse y-partial derivative $\Rightarrow 3x^2y - y^3 - 2y^2 + A(x) = v$

$v_x = 6xy - 0 - 0 + A'(x) = -u_y$ (CRE)

$u_y = 0 - 6xy - 4x + 0 \Rightarrow -u_y = 6xy + 4x$

$6xy + A'(x) = 6xy + 4x$

$A'(x) = 4x$

$A(x) = 2x^2 + C$

$(x+iy)^2 = x^2 - y^2 + 2xyi$

$(x+iy)^3 = x^3 - 3xy^2 - y^3 i + 3x^2yi$

$v = 3x^2y - y^3 - 2y^2 + 2x^2 + C$

$u(1,1) = 0 = 1 - 3 - 4 + 6 = 0$ ✓

$v(1,1) = 0 = 3 - 1 - 2 + 2 + C \Rightarrow C = -2$

$f = u + iv = x^3 - 3xy^2 - 4xy + 6 + i(3x^2y - y^3 - 2y^2 + 2x^2 - 2) = 2iz^2 + z^3 + 6 - 2i$

(b) (*4 points*) Confirm that both $v(x,y)$ and its harmonic conjugate $u(x,y)$ are examples of functions of two variables $\phi(x,y)$ that satisfy the 2-dimensional Laplace Equation $\phi_{xx} + \phi_{yy} = 0$.

$u = x^3 - 3xy^2 - 4xy + 6$

$u_x = 3x^2 - 3y^2 - 4y$

$u_{xx} = 6x$

$u_y = -6xy - 4x$

$u_{yy} = -6x$

$u_{xx} + u_{yy} = 6x - 6x = 0$ ✓

$v = 3x^2y - y^3 - 2y^2 + 2x^2 - 2$

$v_x = 6xy + 4x$

$v_{xx} = 6y + 4$

$v_y = 3x^2 - 3y^2 - 4y$

$v_{yy} = -6y - 4$

$v_{xx} + v_{yy} = 6y - 6y - 4 + 4 = 0$ ✓