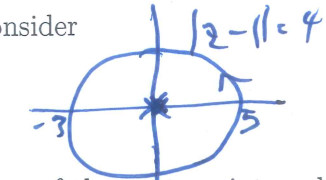
Math 312 Spring 2016 **SHOW ALL YOUR WORK & EXPLAIN EVERY ANSWER** BONUS QUIZ **THREE**

1. Adapted from Question 2 of the **Math 312 Spring 1998 Final Exam**. Consider



$$\oint_{|z-1|=4} z^n e^{1/z}\, dz$$

where n is any integer. The point of this quiz is to derive a formula for the value of the contour integral involving all integer values of n.

(a) *[2 points.]* Consider $n \geq 0$. Derive a formula which evaluates this integral. Explain the method you are using and why you chose this method.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots + \frac{z^n}{n!}$$

$$e^{1/z} = 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2\frac{1}{2!} + \left(\frac{1}{z}\right)^3\frac{1}{3!} + \dots + \left(\frac{1}{z}\right)^n\frac{1}{n!}$$

$$z^n e^{1/z} = z^n + z^{n-1} + z^{n-2}\left(\frac{1}{2!}\right) + z^{n-3}\left(\frac{1}{3!}\right) + \dots + \frac{1}{n!} + \frac{1}{(n+1)!}\frac{1}{z} + \dots$$

$\mathrm{Res}(z^n e^{1/z}, 0) =$ coeff of $\frac{1}{z}$ term

$$\int_{|z-1|=4} e^{1/z} z^n\, dz = 2\pi i \cdot \frac{1}{(n+1)!}$$

by Cauchy Residue Theorem

(b) *[1 point.]* Consider $n < 0$. Derive a formula which evaluates this integral. Explain the method you are using and why you chose this method.

There will only be a $\frac{1}{z}$ term when $n=-1$ or $m=1$

If $n<0$ let $m=-n$

$$\frac{e^{1/z}}{z^m} = z^n e^{1/z} = \frac{1}{z^m}\left(1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2\frac{1}{2!} + \dots\right)$$

$$= \frac{1}{z^m} + \frac{1}{z^{m+1}} + \frac{1}{z^{m+2}}\frac{1}{2!} + \dots$$

$$\mathrm{Res}(z^n e^{1/z}, 0) = \begin{cases} 1, & n=-1 \\ 0, & n<-1 \end{cases}$$

$$\oint z^n e^{1/z}\, dz = \begin{cases} 2\pi i, & n=-1 \\ 0, & n<-1 \end{cases}$$

(c) *[1 point.]* In order to verify your previously derived formula (found in part (b)), write down the value of $\oint_{|z-1|=4} \frac{e^{1/z}}{z^4}\, dz$.

$m=4$ or $n=-4$,

so

$$\oint_{|z-1|=4} \frac{e^{1/z}}{z^4}\, dz = 0$$

(d) *[1 point.]* In order to verify your previously derived formula (found in part (a)), write down the value of $\oint_{|z-1|=4} z^3 e^{1/z}\, dz$.

$m=-3$ or $n=3$

$$\oint_{|z-1|=4} z^3 e^{1/z}\, dz = \frac{2\pi i}{(3+1)!} = \frac{2\pi i}{24} = \frac{\pi i}{12}$$