Jacobi Elliptic Functions and the Classical Pendulum

Kayla Currier

April 19, 2016

Abstract

Elliptic functions are meromorphic functions in the complex plane with two periods that have a positive imaginary ratio. They have three basic forms that come from inversions of elliptic integrals of the first, second, and third kind. The inversions of elliptic integrals of the first kind are known as Jacobi elliptic functions. These functions, introduced by Mathematician Carl Jacobi, are directly related to the Law of Quadratic Reciprocity and have analogies to trigonometric functions. A general description of Jacobi elliptic function will be given along with the derivation of how they are found from the related elliptic integral. The discussion of these functions will involve their application to Physics in the equation of the pendulum. The elliptic integral of the first kind is used to solve this equation for amplitudes larger than what the small angle approximation can handle. In this case one period is real-valued while the other is imaginary. The interpretation of this solution is useful in many problems in classical mechanics.

Outline

I will begin by giving definitions of terms needed to describe elliptic functions and integrals. These include meromorphic functions, poles, and elliptic functions which satisfy

$$f(z+2\omega_1) = f(z+2\omega_2) = f(z)$$

I will then discuss the derivation of the Jacobi functions from the elliptic integral of the first kind:

$$u = \int_0^{\phi} \frac{dt}{\sqrt{1-k^2 sin^2 t}}$$

After I will introduce the differential equation problem of the pendulum and describe how elliptic functions are used in the solution.

Works Cited

1. Bowman, Frank. *Introduction to Elliptic Functions, with Applications*. New York: Dover Publications, 1961. Print.
2. Chandrasekharan, K. *Elliptic Functions*. Berlin: Springer-Verlag, 1985. Print.
3. Gowers, Timothy, June Barrow-Green, and Imre Leader. *The Princeton Companion to Mathematics*. Princeton, NJ: Princeton UP, 2008. Print.
4. Weisstein, Eric W. "Elliptic Function." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/EllipticFunction.html