Multivariable Calculus

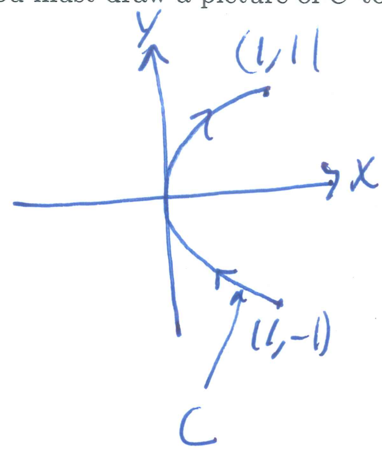
11. Evaluate the line integral

$$\int_C (y - x)\, dx + x^2 y\, dy$$

where C is the curve along $y^2 = x$ from $(1, -1)$ to $(1, 1)$. [HINT: You must draw a picture of C to get full credit.]

$y = t, \ -1 < t < 1$

$x = t^2, \ -1 < t < 1$



$\frac{dy}{dt} = 1 \qquad \frac{dx}{dt} = 2t$

$\int_{-1}^{1} (t - t^2)\, 2t \cdot dt + (t^2)^2 t\, 1 \cdot dt$

$= \int_{-1}^{1} 2t^2 - 2t^3 + t^5\, dt$

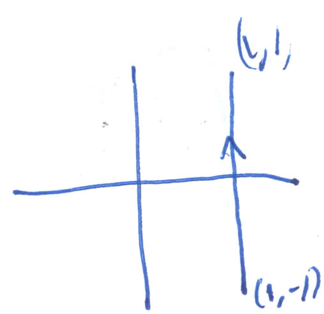
$= \frac{2t^3}{3} - \frac{2t^4}{4} + \frac{t^6}{6} \Big|_{-1}^{1}$

$= \frac{2(1)^3}{3} - \frac{2(1)^4}{4} + \frac{(1)^6}{6} - \left[\frac{2(-1)^3}{3} - \frac{2(-1)^4}{4} + \frac{(-1)^6}{6}\right]$

$= \frac{2}{3} + \frac{2}{3}$

$\boxed{I_C = \frac{4}{3}}$

D



$\frac{dx}{dt} = 0, \ \frac{dy}{dt} = 1$

$x = 1, \ y = t, \ -1 \le t \le 1$

$\int_{-1}^{1} (t - 1) \cdot 0 + 1 \cdot t\, dt = \int_{-1}^{1} t\, dt = \frac{t^2}{2}\Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$

$\boxed{I_D = 0}$

12. Given $f(x, y, z) = \sin(2y + z) + e^{-x+y} + 2\sqrt{x + z + 1}$, find the rate of change of f in the direction $\vec{u} = 4\hat{\imath} + 3\hat{k}$ at the origin, i.e. $D_{\vec{u}} f(0, 0, 0)$.

$$\vec{\nabla} f = f_x \hat{\imath} + f_y \hat{\jmath} + f_z \hat{k}$$

$$f_x = 0 - 1 \cdot e^{-x+y} + \frac{1}{\sqrt{x+z+1}} \qquad f_x(0,0,0) = 0 - 1 + 1 = 0$$

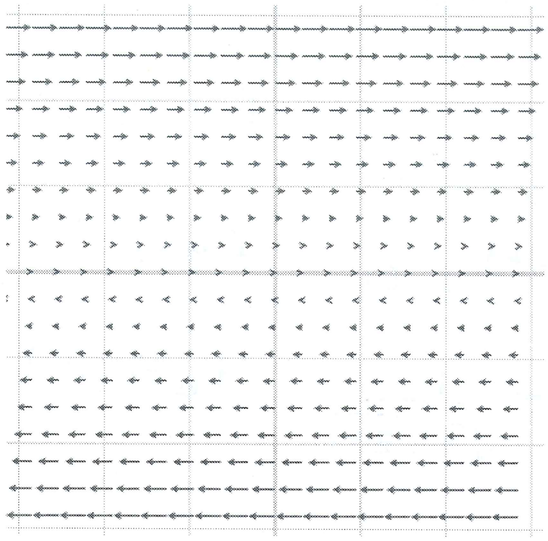
$$f_y = 2\cos(2y + z) + 1 \cdot e^{x+y} + 0 \qquad f_y(0,0,0) = 2 + 1 + 0 = 3$$

$$f_z = \cos(2y + z) + 0 + \frac{1}{\sqrt{x+z+1}} \qquad f_z(0,0,0) = 1 + 0 + 1 = 2$$

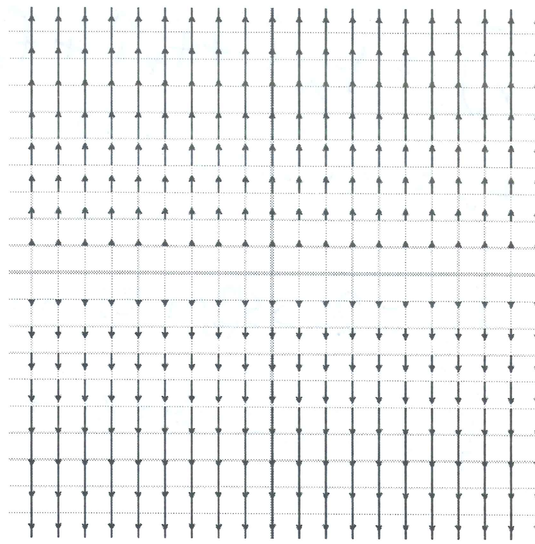
$$\vec{\nabla} f(0,0,0) = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \qquad \hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = (4\hat{\imath} + 3\hat{k})\frac{1}{5}$$

$$D_{\vec{u}} f(0,0,0) = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} \frac{1}{5} = 0 + 0 + \frac{6}{5}$$

13. Consider the two vector fields depicted in the figure below, labelled Field $\vec{A}$ and Field $\vec{B}$.



FIELD $\vec{A}$



FIELD $\vec{B}$

One of these two vector fields is a conservative field, i.e. can be written as the gradient of some function $\vec{\nabla} f$. Identify the conservative field, explain your selection and find the potential function $f(x, y)$ which produces the vector field you have selected.

$$\vec{\nabla} \times \vec{A} = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \quad \text{where } \vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} = cy\hat{\imath},\ c \neq 0$$

$$\vec{\nabla} \times \vec{A} = -c$$

$$\vec{\nabla} \times \vec{B} = \frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} \quad \text{where } \vec{B} = B_1\hat{\imath} + B_2\hat{\jmath} = dy\hat{\jmath},\ d \neq 0$$

$$= 0 - 0 = 0 \Rightarrow \vec{B} = \vec{\nabla} f$$

$$\vec{\nabla} f = f_x\hat{\imath} + f_y\hat{\jmath} = y\hat{\jmath}$$

$f_x = 0 \qquad f = c + \phi(y)$

$f_y = y \qquad f = \frac{y^2}{2} + c$

$$f = \frac{y^2}{2} + c$$

$$\vec{\nabla} f = y\hat{\jmath} + 0\hat{\imath}$$

$$\vec{A} = \vec{\nabla} f = y\hat{\imath} + 0\hat{\jmath} = f_x\hat{\imath} + f_y\hat{\jmath}$$

$f_x = y \Rightarrow f = xy + \phi(x)$

$f_y = 0 \qquad f_y = x + 0 \neq 0$ No such f

$\vec{B}$ is a gradient field and $\vec{A}$ is not!

14. Show that the surfaces

$$z = \sqrt{x^2 + y^2} \quad = f_1$$

and

$$z = \frac{1}{10}(x^2 + y^2) + \frac{5}{2} \quad = f_2$$

intersect at $(3, 4, 5)$ and have a common tangent plane at that point and find it's equation.

Check intersection

$$f_1(3,4) = \sqrt{3^2+4^2} = \sqrt{25} = 5 \checkmark$$

$$f_2(3,4) = \frac{1}{10}(3^2+4^2) + \frac{5}{2} = \frac{25}{10} + \frac{5}{2} = 5 \checkmark$$

Check gradient

$$\vec{\nabla} f_1 = f_x \hat{\imath} + f_y \hat{\jmath}$$

$$= \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x\hat{\imath} + \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y\hat{\jmath}$$

$$\vec{\nabla} f_1(3,4) = \frac{3}{\sqrt{3^2+4^2}}\hat{\imath} + \frac{4}{\sqrt{3^2+4^2}}\hat{\jmath} = \frac{3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath}$$

$$\vec{\nabla} f_2 \cancel{(x,y)} = \frac{1}{10}2x\hat{\imath} + \frac{1}{10}2y\hat{\jmath}$$

$$\vec{\nabla} f_2(3,4) = \frac{6}{10}\hat{\imath} + \frac{8}{10}\hat{\jmath} = \frac{3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath}$$

$$z - 5 = \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$$

$$z = \frac{3x}{5} - \frac{9}{5} + \frac{4y}{5} - \frac{16}{5} + 5$$

$$z = \frac{3x}{5} + \frac{4y}{5}$$

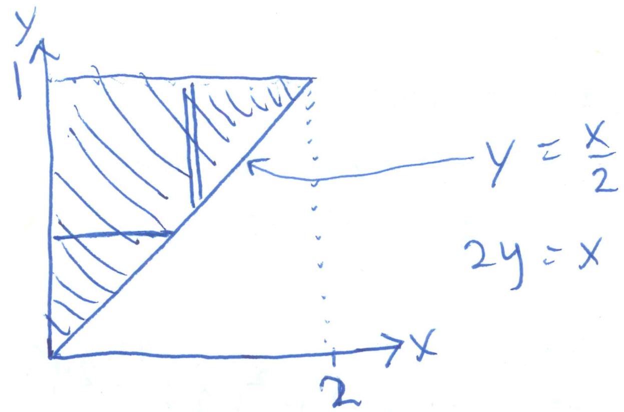
$$5z - 3x + 4y = 0$$

15. Consider the following iterated integral

$$\mathcal{I} = \int_0^2 \int_{x/2}^1 e^{y^2}\, dy\, dx$$

(a) Sketch a picture of the region being integrated over in the xy-plane.

(b) Reverse the order of integration to obtain the exact value of $\mathcal{I}$.



$$\mathcal{I} = \int_0^1 \int_0^{2y} e^{y^2}\, dx\, dy$$

$$= \int_0^1 2y e^{y^2}\, dy$$

$$= e^{y^2}\Big|_0^1 = e^1 - e^0 = e - 1$$