PRACTICE Comprehensive Exam **Student Number (AXXXXXXX):** SOLNS

Department of Mathematics Tuesday, March 15, 2016

Closed book. Closed notes. NO CALCULATORS. Time allowed: 3 hours for 5 sections (proportionally less if taking fewer than 5 sections). In other words, 36 minutes for each section taken. Please write very legibly and cross out all scratch work.

Calculus 1

1. ______ 2. ______ 3. ______ 4. ______ 5. ______ Total: ______

Calculus 2

6. ______ 7. ______ 8. ______ 9. ______ 10. ______ Total: ______

Multivariable Calculus

11. ______ 12. ______ 13. ______ 14. ______ 15. ______ Total: ______

Linear Algebra

16. ______ 17. ______ 18. ______ 19. ______ 20. ______ Total: ______

Discrete Mathematics

21. ______ 22. ______ 23. ______ 24. ______ 25. ______ Total: ______

Calculus 2

1. Consider the sequence $\{a_n\}_{n=1}^{+\infty}$ whose nth term is

$$a_n = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+(k/n)}.$$

Show that $\lim_{n\to+\infty} a_n = \ln 2$ by interpreting a_n as the Riemann sum of a definite integral.

$f(x) = \frac{1}{1+x}$, $\Delta x = \frac{1}{n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{n} f(k\Delta x) = \int_0^1 \frac{1}{1+x}\,dx$$

This represents a righthand Riemann sum of $f(x) = \frac{1}{1+x}$ on interval $0 \le x \le 1$ with $\Delta x = 1/n$ as $n \to \infty$

$$= \ln(1+x)\Big|_0^1$$
$$= \ln 2 - \ln 1$$
$$= \ln 2$$

2. Use the integral test to investigate the relationship between the value of p and the convergence of the series

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}.$$

$u = \ln k \quad du = \frac{1}{k}dk$

$$\int_2^{\infty} \frac{1}{k(\ln k)^p}\,dk = \int_{\ln 2}^{\infty} \frac{1}{u^p}\,du = \lim_{b\to\infty} \int_{\ln 2}^{b} \frac{1}{u^p}\,du = \lim_{b\to\infty} \frac{u^{-p+1}}{-p+1}\Big|_{\ln 2}^{b} \quad (p \ne 1)$$

$$= \lim_{b\to\infty} \frac{b^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} = \frac{(\ln 2)^{p-1}}{p-1} \qquad 1-p<0 \text{ i.e } p>1$$

If $p = 1$

$$\int_{\ln 2}^{\infty} \frac{1}{u}\,du = \lim_{b\to\infty} \int_{\ln 2}^{b} \frac{du}{u} = \lim_{b\to\infty} \ln u\Big|_{\ln 2}^{b} = \lim_{b\to\infty} \ln b - \ln(\ln 2)$$
$$= \infty$$

$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$ converges when $p>1$ and diverges when $p \le 1$

3. Determine whether the following series converges or diverges

$$\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}.$$

(Of course explain your work and cite any theorems (convergence tests) you use and why the given series satisfies the hypotheses of those theorems (convergence tests).)

With factorials Absolute Ratio Test usually a good idea

Absolute Ratio Test

$$L = \lim_{K\to\infty} \left|\frac{a_{K+1}}{a_K}\right| \Leftrightarrow \begin{array}{l} L < 1, \ \sum a_K \text{ CONVERGES} \\ L > 1, \ \sum a_K \text{ DIVERGES} \\ L = 1, \ \text{NO CONCLUSION} \end{array}$$

$$a_K = \frac{(K!)^2}{(2K)!} \qquad a_{K+1} = \frac{[(K+1)!]^2}{(2K+2)!} \qquad a_K > 0 \text{ for all } K$$

$$\left|\frac{a_{K+1}}{a_K}\right| = \frac{a_{K+1}}{a_K} = \frac{\dfrac{(K+1)!\,(K+1)!}{(2K+2)!}}{\dfrac{K!\,K!}{(2K)!}} = \frac{(K+1)!\,(K+1)!\,(2K)!}{K!\ \ K!\,(2K+2)!}$$

$$= \frac{(K+1)K!\,(K+1)K!\,(2K)!}{K!\ \ K!\,(2K+2)(2K+1)(2K)!} = \frac{(K+1)(K+1)}{(2K+2)(2K+1)}$$

$$\lim_{K\to\infty} \frac{a_{K+1}}{a_K} = \lim_{K\to\infty} \frac{(K+1)(K+1)}{(2K+2)(2K+1)} = \lim_{K\to\infty} \frac{1+\frac{2}{K}+\frac{1}{K^2}}{4+\frac{6}{K}+\frac{2}{K^2}}$$

$$= \frac{1}{4} < 1$$

Thus $\displaystyle\sum_{K=0}^{\infty} \frac{(K!)^2}{(2K)!}$ converges by the absolute ratio test

4. Consider the following reduction formula (which is valid for all $n \geq 1$):

$$\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx$$

(a) Use integration by parts to derive the reduction formula.

(b) Use the formula to obtain an integral-free expression for the $n = 3$ case, i.e. simplify $\int (\ln x)^3 \, dx$.

$$\int u\,dv = uv - \int v\,du$$

$u = (\ln x)^n \qquad du = n(\ln x)^{n-1} \cdot \frac{1}{x}\,dx$

$dv = dx \qquad v = x$

$$\int (\ln x)^n dx = x \cdot (\ln x)^n - \int x \cdot n (\ln x)^{n-1} \cdot \frac{1}{x}\,dx$$

$$= x \cdot (\ln x)^n - \int n (\ln x)^{n-1} dx$$

$n = 3$

$$\int (\ln x)^3 dx = x \cdot (\ln x)^3 - 3 \int (\ln x)^2 dx$$

$$\int (\ln x)^2 dx = x (\ln x)^2 - 2 \int \ln x \, dx$$

$$\int \ln x \, dx = x \ln x - 1 \cdot \int dx = x \ln x - x$$

$$\int (\ln x)^3 dx = x \cdot (\ln x)^3 - 3\left[x \cdot (\ln x)^2 - 2 \int \ln x \, dx\right]$$

$$= x \cdot (\ln x)^3 - 3x(\ln x)^2 + 6 \int \ln x \, dx$$

$$= x \cdot (\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x + C$$

5. The definite integral

$$\mathcal{I} = \int_3^8 \frac{x}{\sqrt{x+1}}\, dx$$

represents an area and a net change.

(a) The integral $\mathcal{I}$ represents the **area** of what? (HINT: provide a sketch!)

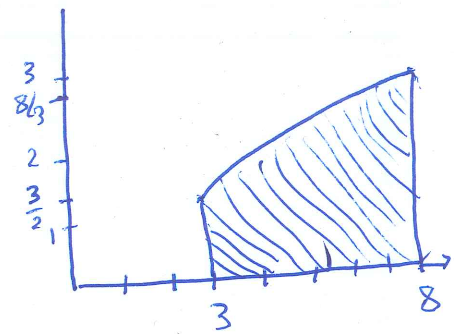
(b) The integral $\mathcal{I}$ represents the **net change** of what?

(c) Evaluate $\mathcal{I}$ exactly.

1pt a. I represents the area bounded by $x=3,\ x=8,\ y=0$ and $y=\frac{x}{\sqrt{x+1}}$

$f(x)=\frac{x}{\sqrt{x+1}}$ $\quad f(3)=\frac{3}{\sqrt{4}}=\frac{3}{2}$

$f(8)=\frac{8}{\sqrt{9}}=\frac{8}{3}$



1pt b. $I=\int_3^8 \frac{x}{\sqrt{x+1}}\,dx = F(8)-F(3)$

where $F'(x)=\frac{x}{\sqrt{x+1}}$

I represents the net change of $F(x)$ from 3 to 8.

2pts c. $I=\int_3^8 \frac{x}{\sqrt{x+1}}\,dx = \int_4^9 \frac{u-1}{\sqrt{u}}\,du = \int_4^9 u^{1/2}-u^{-1/2}\,du$

$u=x+1$ $\quad x=3,\ u=4$

$du=dx$ $\quad x=8,\ u=9$

$I = \frac{2}{3}u^{3/2}-2u^{1/2}\Big|_4^9$

$I=\left(\frac{2}{3}9^{3/2}-2\sqrt{9}\right)-\left(\frac{2}{3}4^{3/2}-2\sqrt{4}\right)$

$I=\left(\frac{2}{3}\cdot 3^3-2\cdot 3\right)-\left(\frac{2}{3}\cdot 2^3-2\cdot 2\right)$

$=(2\cdot 3^2-6)-\left(\frac{16}{3}-4\right)$

$=(18-6)-\left(\frac{16}{3}-\frac{12}{3}\right)$

$=12-\frac{4}{3}=\frac{36-4}{3}=\frac{32}{3}$