BUCKMIRE

MATHEMATICS COMPREHENSIVE EXAMINATION, SPRING 2006

Linear Systems

16. a. Prove or disprove: If λ is an eigenvalue of A, then λ^m is an eigenvalue of A^m.

    b. Find all the eigenvalues of the permuation matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Base step

a. $A\vec{x} = \lambda\vec{x}$

$A(A\vec{x} = A(\lambda\vec{x})$

$A^2\vec{x} = \lambda(A\vec{x}) = \lambda\lambda\vec{x} = \lambda^2\vec{x}$

Inductive step

$A^n\vec{x} = \lambda^n\vec{x} \Rightarrow A^{n+1}\vec{x} = \lambda^{n+1}\vec{x}$

$AA^n\vec{x} = A\lambda^n\vec{x} =$

$A^{n+1}\vec{x} = \lambda^n(A\vec{x}) = \lambda^n\lambda\vec{x} = \lambda^{n+1}\vec{x}$

(b) $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

eigenvalues of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are 1 and -1.

17. This problem concerns subspaces of a vector space $\mathbf{R}^2$.

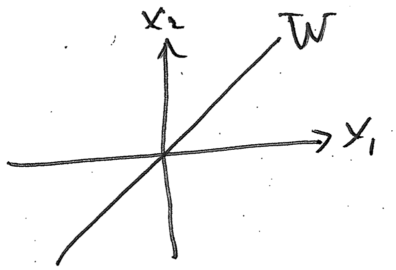
    a. Prove or disprove: $V = \{(0,1) + c(1,1),\ c \in \mathbf{R}\}$ is a subspace of $\mathbf{R}^2$.

    b. Find a subpace of $\mathbf{R}^2$ *orthogonal* to the subspace $W = \{c(1,1),\ c \in \mathbf{R}\}$.

a. V is NOT a subspace of $\mathbb{R}^2$ since $\vec{0} \notin V$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ 1+c \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for any value of $c \in \mathbb{R}$.

(b)



$W^\perp$ is the subspace $W^\perp = \left\{d\begin{pmatrix} 1 \\ -1 \end{pmatrix}, d \in \mathbb{R}\right\}$

$d\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} c = cd \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = cd \cdot 0$
$= 0$ ✓

18. a. Define what it means for a finite set of vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ to be *linearly independent*.

b. Is the following set of ~~three~~ four vectors in $\mathbf{R}^2$ linearly independent? Explain your answer.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

a. A finite set of vectors is linearly independent if the ONLY solution to $\sum_{i=1}^{n} c_i \vec{v_i} = \vec{0}$ is $c_i = 0$ for all $i = 1$ to n

b. $\begin{pmatrix} 1 & 3 & 2 & 2 \\ 2 & 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$0\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 3 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 2 \\ -2 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

These are Linearly dependent vectors in $\mathbb{R}^2$

$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 2 & 0 \\ 2 & 0 & -2 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 3 & 2 & 2 & 0 \\ 0 & -6 & -6 & -3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 3 & 2 & 2 & 0 \\ 0 & 1 & 1 & 1/2 & 0 \end{array}\right)$

$\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1/2 & 0 \\ 0 & 1 & 1 & 1/2 & 0 \end{array}\right)$

~~$c_1 = 0 \quad c_3 = \frac{1}{2} c_4$~~

~~$c_2 = 0 \quad c_3 = \frac{1}{2} c_4$~~

$\Rightarrow$ ~~$c_3 = c_4 = 0$~~

$c_1 - c_3 + \frac{1}{2} c_4 = 0$

$c_2 + c_3 + \frac{1}{2} c_4 = 0$

c_3 and c_4 can have any value

Let $c_4 = 2$

$c_3 = 1$

$\Rightarrow c_1 = 0$

$c_2 = -2$

19. Use the fact that $\det(AB) = \det(A)\det(B)$ to prove:

If A is nonsingular, then $\det(A^{-1}) = 1/\det(A)$.

$AA^{-1} = I$

$\det(AA^{-1}) = \det(I)$

$\det(A) \cdot \det(A^{-1}) = \det(I)$

$\det(A^{-1}) = \dfrac{1}{\det(A)}$

$\det(A) \neq 0$ since A^{-1} exists.

$\det(I) = 1$

20. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 0 & 6 & 10 \end{bmatrix}$.

a. Find a *basis* for the nullspace of A.

b. Either find the general solution of $A\vec{x} = \vec{b}$, where $\vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, or explain why such a solution doesn't exist.

a. $\begin{pmatrix} 1 & 2 & 2 & 4 \\ 3 & 0 & 6 & 10 \end{pmatrix} \xrightarrow{R_2' = R_2 - 3R_1} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & -6 & 0 & -2 \end{pmatrix} \xrightarrow{R_2' = R_2(-1/6)} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 1/3 \end{pmatrix} \xrightarrow{R_1' = R_1 - 2R_2} \begin{pmatrix} 1 & 0 & 2 & 10/3 \\ 0 & 1 & 0 & 1/3 \end{pmatrix}$

Basis for the nullspace of A is $\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -10/3 \\ -1/3 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\begin{pmatrix} 1 & 2 & 2 & 4 \\ 3 & 0 & 6 & 10 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ✓

$\begin{pmatrix} 1 & 2 & 2 & 4 \\ 3 & 0 & 6 & 10 \end{pmatrix} \begin{pmatrix} -10 \\ -1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -10 - 2 + 12 \\ -30 + 30 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ✓

b. $\text{col}(A) = \mathbb{R}^2$ so $\vec{b} \in \text{col}(A) \Rightarrow$ a solution must exist to $A\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 4 & 1 \\ 3 & 0 & 6 & 10 & 2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 2 & 4 & 1 \\ 0 & -6 & 0 & -2 & -1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 2 & 4 & 1 \\ 0 & 1 & 0 & 1/3 & 1/6 \end{array}\right) \to$

$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 10/3 & 2/3 \\ 0 & 1 & 0 & 1/3 & 1/6 \end{array}\right)$ $\qquad \vec{x} = \begin{pmatrix} 2/3 \\ 1/6 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} -10 \\ -1 \\ 0 \\ 3 \end{pmatrix}$