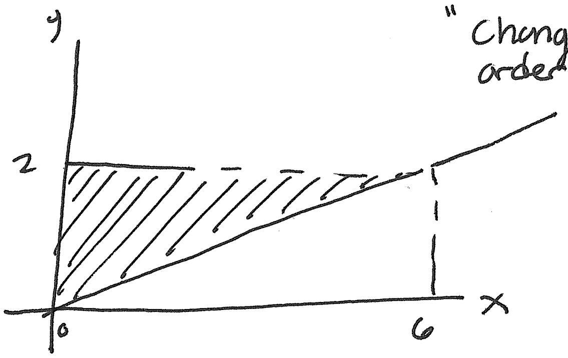
MULTIVARIABLE/VECTOR CALC 2005

11) $$\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\,dy\,dx = \int_0^2 \int_0^{3y} x\sqrt{y^3+1}\,dx\,dy$$

"Change order of integration."

$y = x/3$



$$= \int_0^2 \sqrt{y^3+1}\left(\int_0^{3y} x\,dx\right)dy$$

$$= \int_0^2 \sqrt{y^3+1}\left(\tfrac{1}{2}x^2\Big|_0^{3y}\right)dy$$

$$= \frac{1}{2}\int_0^2 (3y)^2\sqrt{y^3+1}\;dy$$

$$= \frac{3}{2}\int_0^2 3y^2\sqrt{y^3+1}\;dy$$

u-substitution

$u = y^3+1$

$du = 3y^2dy$

$$= \frac{3}{2}\int_0^2 u^{1/2}\,du$$

$$= \frac{\cancel{3}}{\cancel{2}}\left(\frac{\cancel{2}}{\cancel{3}}u^{3/2}\Big|_0^2\right)$$

$$= (y^3+1)^{3/2}\Big|_0^2$$

$$= (2^3+1)^{3/2} - (0^3+1)^{3/2}$$

$$= 27 - 1 = \boxed{26}$$

12) (a) $\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$ $\qquad f(x,y) = 4 - x^2 - 2y^2$

$$= (-2x, -4y)$$

At given pt. $(1,-1,1)$ $\Rightarrow$ $\vec{\nabla} f(1,-1,1) = (-2(1), -4(-1)) = \boxed{(-2, 4)}$

gradient gives direction of greatest rate of change.

(b) Along the direction $\vec{v} = 3\hat{i} + 4\hat{j}$

$$= \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\|\vec{v}\| = \sqrt{3 \cdot 3 + 4 \cdot 4} = \sqrt{9 + 16} = \sqrt{25}$$
$$= 5$$

normalized vector $\vec{u} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$

Dot product of $\vec{\nabla} f \cdot \vec{u}$ gives rate of change.

$$\vec{\nabla} f(1,-1,1) \cdot \vec{u} = (-2, 4) \cdot (3/5, 4/5)$$
$$= -6/5 + \frac{16}{5} = 10/5 = \boxed{2}$$

15) $\vec{F}(x,y) = (6xy + y^2,\ 3x^2 + 2xy)$

over line segment $(1,2)$ to $(-1,-2)$.

$\Rightarrow$ formula of line through points

$$m = \frac{2+2}{1+1} = 2 \qquad y = 2x + b \qquad \boxed{y = 2x}$$

$$2 = 2(1) + b$$

$$-2 \quad -2$$

$$\cancel{y}\ b = 0$$

So... $\vec{r} = (x,y) = \begin{pmatrix} x \\ 2x \end{pmatrix}$

$$d\vec{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} dx$$

we find $\vec{F}(x,y) = \left(6x(2x) + (2x)^2,\ 3x^2 + 2x(2x)\right)$

$$= (12x^2 + 4x^2,\ 3x^2 + 4x^2)$$

$$= (16x^2,\ 7x^2)$$

$$\oint_c \vec{F} \cdot d\vec{r} = \int_{-1}^{1} (16x^2, 7x^2) \cdot (x, 2x)\, dx$$

$$= \int_{-1}^{1} (16x^3 + 14x^3)\, dx$$

$$= 30 \int_{-1}^{1} x^3\, dx = \frac{30}{4} x^4 \Big|_{-1}^{1} = \frac{30}{4}\left(1^4 - (-1)^4\right)$$

$$= \boxed{0}.$$

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Math 300
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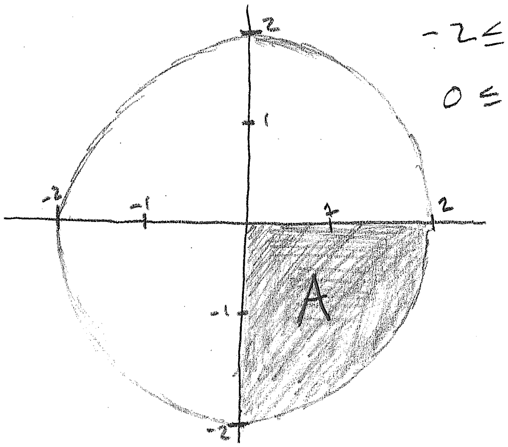
] Sketch the region of integration, and evaluate the integral:

$$\int_{-2}^{0}\int_{0}^{\sqrt{4-y^2}} e^{-x^2-y^2}\,dx\,dy$$



$-2 \le y \le 0$

$0 \le x \le \sqrt{4-y^2}$

note: Polar Coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$^2+y^2 = r^2$

$^2-y^2 = -r^2$

we can see that we must change to polar coordinates because there is no antiderivative of e^{-x^2}:

our integral becomes,

$$\int_{-\pi/2}^{0}\int_{0}^{2} e^{-r^2} r\,dr\,d\theta$$

we use U substitution:

$u = r^2 \qquad du = -2r\,dr$

$dr = \dfrac{du}{-2r}$

we evaluate our inner integral first

$$\int_0^2 e^u \cancel{r}\frac{du}{-2\cancel{r}} = -\frac{1}{2}\int_0^2 e^u = -\frac{1}{2}e^{-r^2}\Big|_0^2$$

$$= -\frac{1}{2}\left(e^{-4}+1\right)$$

now we evaluate our outer integral

$$\int_{-\pi/2}^{0} -\frac{1}{2}\left(e^{-4}+1\right)d\theta = -\frac{1}{2}\int_{-\pi/2}^{0} e^{-4}+1\,d\theta =$$

$$= -\frac{1}{2}\left[\left(e^{-4}+1\right)\theta\right]\Big|_{-\pi/2}^{0}$$

$$= -\frac{1}{2}\left[\theta e^{-4}+\theta\right]\Big|_{-\pi/2}^{0}$$

$$= -\frac{1}{2}\left[\left(\cancel{0e^{-4}}^{\,0}+0\right)-\left(-\tfrac{\pi}{2}e^{-4}-\tfrac{\pi}{2}\right)\right]$$

$$= -\frac{1}{2}\left(\frac{\pi}{2}e^{-4}+\frac{\pi}{2}\right) = \left|-\frac{\pi}{4}e^{-4}-\frac{\pi}{4}\right| = A$$

2) ~~f~~ Find the absolute (global) maximum and minimum of $f(x,y) = 2x^2 + 2xy + y^2 + 2x$ on the region $2x^2 + 2xy + y^2 \leq 2$ (This is the region inside an ellipse whose boundary includes values $-\sqrt{2} \leq x \leq \sqrt{2}$.)

2)

We know that because $F(x,y)$ is continuous on a closed and bounded set, it has both global maximum and global minimum.

we can see that because our x is bounded there will be a maximum where x is largest: so where $x = \sqrt{2}$ we have $F(\sqrt{2},y) = 2\sqrt{2}^2 + 2\sqrt{2}\,y + y^2 + 2\sqrt{2} \leq$

$2 + 2\sqrt{2}$

$\boxed{\text{we have global max where } x = \sqrt{2}}$

for our global minimum we can see it will be within the bounded region so we first find critical points of $f(x,y)$ by finding partial derivatives $f_x(x,y)$ and $f_y(x,y)$: set them both equal to zero, evaluate for x and y.

$f_x(x,y) = 4x + 2y + 2$ $\qquad$ $f_y(x,y) = 2x + 2y$

bstitute $:= -y$

$4x + 2y + 2 = 0$ $\qquad$ $2x + 2y = 0$

$-4y + 2y + 2 = 0$ $\qquad$ $x + y = 0$

$2y = 2$ $\qquad$ $x = -y$

$y = 1$ ✓

$x = -1$

$\boxed{\text{Our minimum will be when } x = -1 \text{ and } y = 1}$

$$\min F(x,y) = f(-1,1) = 2(-1)^2 + 2(-1)(1) + (1)^2 + 2(-1)$$

$$= (2 + (-2)) + 1 + (-2)$$

$$0$$

$$= 1 + -2 = -1$$

\# 13

Find an equation for the plane through the point $(1,2,3)$ that is parallel to the vectors $\vec{u} = (1,0,4)$ and $\vec{v} = (-2,1,2)$

First we find a vector orthogonal to $\vec{u}$ and $\vec{v}$; in other words we find the normal vector.

We use the cross product defined as:

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2)i - (u_1 v_3 - u_3 v_1)j + (u_1 v_2 - u_2 v_1)k$$

So with our vectors $\vec{u}$ and $\vec{v}$ we have

$$\vec{u} \times \vec{v} = ((0 \cdot 2) - (4 \cdot 1))i - ((1 \cdot 2) - (4 \cdot -2))j + ((1 \cdot 1) - (0 \cdot -2))k = -4j - 10j + 1k$$

$\therefore$ our normal vector is $\vec{n} = (-4, -10, 1)$

we test for orthogonality with our vectors $\vec{u}$ and $\vec{v}$

$$\begin{pmatrix} -4 \\ -10 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = -4 + 0 + 4 = 0 \checkmark$$

$$\begin{pmatrix} -4 \\ -10 \\ 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = 8 - 10 + 2 = 0 \checkmark$$

now we can use point normal form to write an equation for our plane where $\vec{n} = (-4, -10, 1)$ and $P = (1,2,3)$

So our plane is represented by: $-4(x-1) + -10(y-2) + 1(z-3) = 0$

14] Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = (y, x+y)$, and C is the arc of the parabola $x = y^2$ from $(0,0)$ to $(4,-2)$

→ we find $\vec{r} = (x,y) = (y^2, y)$ because C is the parabola $x = y^2$

→ now we find $d\vec{r} = (2y, 1)\,dy$

→ we find $F(x,y) = (y, x+y) = (y, y^2+y)$ because $x = y^2$

now $\int_C \vec{F} \cdot d\vec{r} = \int (y, y^2+y) \cdot (2y, 1)\,dy =$

$\int (2y^2) + (y^2+y)\,dy =$ with dot product

[w]e integrate over [o]ur y's from [(0],0) and (4,-2)

$\int_0^{-2} 3y^2 + y\,dy = \left[y^3 + \frac{y^2}{2}\right]\Big|_0^{-2} =$

$= \left(-8 + \frac{4}{2}\right) - (0) = -8 + 2 = \boxed{-6}$

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[1]5] Sketch the contour diagram for $f(x,y) = x^2 + y^2 - 1$, with level curves labeled $f=0$, $f=3$, $f=8$. Also label and mark units on your axes appropriately.

we know $f(x,y) = x^2+y^2-1$ is a circle

we are looking for

$f=0$, so $x^2+y^2-1=0$
$x^2+y^2=1$ Circle radius $=1$

$f=3$, so $x^2+y^2-1=3$
$x^2+y^2=4$ Circle radius$=2$

$f=8$, so $x^2+y^2-1=8$
$x^2+y^2=9$ circle radius $=3$