DISCRETE MATH 2004

21) Coefficient of a^3b^5 in $(2a-b)^8$

$$(2a-b)^8 = \sum_{i=0}^{8} \binom{8}{i} (2a)^{8-i} (-b)^i$$

so the term $i=5$ gives a^3b^5

$$\binom{8}{5}(2a)^3(-b)^5 = \frac{8!}{5!\,3!}(8a^3)(-b^5)$$

$$= \frac{8\cdot 7\cdot \cancel{6}}{\cancel{3}\cdot \cancel{2}\cdot 1}(-8a^3b^5)$$

$$= 56(-8a^3b^5)$$

$$= -448a^3b^5$$

So the coefficient is $\boxed{-448}$

24) Prove by induction for $n \geq 2$:

$$2 + 6 + 12 + \dots + (n^2-n) = \frac{n(n^2-1)}{3}$$

Base Step - Prove for $n_0 = 2$

$$2 \stackrel{?}{=} \frac{2(4-1)}{3}$$

$$2 \stackrel{?}{=} \frac{6}{3}$$

$$2 \stackrel{\checkmark}{=} 2 \qquad \text{So true for } n=2.$$

Inductive Step

Inductive Hypothesis ~
Assume the statement holds for $n=k$.

$$2+6+12+\dots+(k^2-k)=\frac{k(k^2-1)}{3}$$

Now show statement is true for $n=k+1$

$$\underbrace{2+6+12+\dots+(k^2-k)}+\left[(k+1)^2-(k+1)\right]\stackrel{?}{=}\frac{(k+1)\left[(k+1)^2-1\right]}{3}$$

by inductive hypothesis >

$$\frac{k(k^2-1)}{3}+\frac{3}{3}\left[(k+1)^2-(k+1)\right]\stackrel{?}{=}\frac{(k+1)\left[(k+1)^2-1\right]}{3}$$

$$\frac{k(k^2-1)+3\left[(k+1)^2-(k+1)\right]}{3}\stackrel{?}{=}\frac{(k+1)\left[(k+1)^2-1\right]}{3}$$

$$\frac{k(k+1)(k-1)+3\left[(k+1)^2-(k+1)\right]}{3}\stackrel{?}{=}\frac{(k+1)\left[(k+1)^2-1\right]}{3}$$

$$\frac{(k+1)\left(k(k-1)+3\left[(k+1)-1\right]\right)}{3}\stackrel{?}{=}\frac{(k+1)\left[(k+1)^2-1\right]}{3}$$

$$\frac{(k+1)(k^2-k+3k)}{3}\stackrel{?}{=}\frac{(k+1)\left[(k+1)^2-1\right]}{3}$$

$$\frac{(k+1)(k^2+2k)}{3}+\frac{(k+1)}{3}-\frac{(k+1)}{3}\stackrel{?}{=}\frac{(k+1)\left[(k+1)^2-1\right]}{3}$$

add & subtract $\frac{(k+1)}{3}$ and factor into existing term >

$$\frac{(k+1)\left[k^2+2k+1-1\right]}{3}\stackrel{?}{=}\frac{(k+1)\left[(k+1)^2-1\right]}{3}$$

... factor (k^2+2k+1) into $(k+1)^2$

$$\frac{(k+1)[(k+1)^2-1]}{3} \overset{\checkmark}{=} \frac{(k+1)[(k+1)^2-1]}{3}$$

So by the Base Step and the Inductive Step the statement

$$2+6+12+\dots+(n^2-n)=\frac{n(n^2-1)}{3}$$

is true for all $n \geq 2$. $\square$

Discrete S, Fall, '04

22 Prove the relation ~ is an equiv. rel. on $\mathbb{Z}$: $a \sim b \Leftrightarrow$ 6 divides $(a-b)$

(a)

1. $a \sim a$

6 divides $(a-a)$ yes, every thing divides 0.

2. If $a \sim b$, $b \sim c$ then $a \sim c$

If $a \sim b$ then $\exists k \in \mathbb{Z}$ s.t. $6k = (a-b)$

If $b \sim c$ then $\exists j \in \mathbb{Z}$ s.t. $6j = (b-c)$

$b = 6j + c$

$\rightarrow 6k = a - b \Rightarrow 6k = a - 6j - c$

$\rightarrow 6(k+j) = a - c$

3. $a \sim b$ iff $b \sim a$

$a \sim b$

$\Leftrightarrow \exists k \in \mathbb{Z}: 6k = a - b$

~~let~~ $j = -k$

$\Leftrightarrow 6j = b - a$

(b) $\{(a, a+6k)\}$

Discrete Fall '04

(23) Let A be a set of n elements. The Power set, $P(A)$, of a set A, is the set of all subsets of A.

(a) How many elements are in $P(A)$? Prove it.

For every subset in the set $P(A)$ each element is either in the subset or not, giving you two choices. Using the product rule, we obtain a total of 2^n subsets of A, so $|P(A)| = 2^n$

(b) Prove or disprove: Let A + B be two finite sets. Then $P(A \cup B) = P(A) \cup P(B)$

False

Let $A = \{1\}$. This is finite

Let $B = \{2\}$. " " "

$A \cup B = \{1, 2\}$

$P(A \cup B) = \{\{\emptyset\}, \{1\}, \{2\}, \{1,2\}\}$

$P(A) = \{\{\emptyset\}, \{1\}\}$

$P(B) = \{\{\emptyset\}, \{2\}\}$

$P(A) \cup P(B) = \{\{\emptyset\}, \{1\}, \{2\}\}$

$P(A) \cup P(B) \neq P(A \cup B)$

'04 Exam

25 $[p \to (q \to r)] \leftrightarrow [(p \to q) \to (p \to r)]$

p	q	r	$q \to r$	$p \to q$	$p \to r$
T	T	T	T	T	T
T	T	F	F	T	F
T	F	T	T	F	T
T	F	F	T	F	F
F	T	T	T	T	T
F	T	F	F	T	T
F	F	T	T	T	T
F	F	F	T	T	T

$p \to (q \to r)$	$(p \to q) \to (p \to r)$	$[p \to (q \to r)] \leftrightarrow [(p \to q) \to (p \to r)]$
T	T	T
F	F	T
T	T	T
T	T	T
T	T	T
T	T	T
T	T	T
T	T	T

So true. $\square$