Math 214 Spring 2007 **SHOW ALL YOUR WORK** Quiz Seven

1. Consider the matrix $A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$ and its inverse, $A^{-1} = \begin{bmatrix} -3/4 & 1/2 \\ 1/2 & 0 \end{bmatrix}$.

a. *(2 points).* Find the eigenvalues, λ_1 and λ_2, of A

$$p(\lambda) = \det(A-\lambda I) = \begin{vmatrix} 0-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = -\lambda(3-\lambda) - 4 = -3\lambda + \lambda^2 - 4 = 0$$

$$(\lambda-4)(\lambda+1) = 0 \Rightarrow \lambda_1 = -1 \text{ and } \lambda_2 = 4$$

b. *(4 points).* Find the corresponding eigenvectors $\vec{x}_1$ and $\vec{x}_2$ of A

$\lambda_1 = -1 \quad E_{-1} = \text{null}(A+I) = \text{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$

$(A+I)\vec{x} = \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad x_1 + 2x_2 = 0$, x_2 free $\quad x_1 = -2x_2 \quad \vec{x}_1 = \begin{pmatrix} -2x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} x_2$

$\lambda_2 = 4 \quad E_4 = \text{null}(A-4I) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$

$(A-4I)\vec{x} = \left(\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad 2x_1 - x_2 = 0$, x_2 free $\quad x_2 = 2x_1 \quad \vec{x}_2 = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

[Notice $A = A^t$ so $\vec{x}_1 \cdot \vec{x}_2 = 0$]

c. *(2 points).* Find the eigenvalues, $\hat{\lambda}_1$ and $\hat{\lambda}_2$, of A^{-1}.

$$p(\hat{\lambda}) = \det(A^{-1} - \hat{\lambda} I) = \begin{vmatrix} -3/4-\hat{\lambda} & 1/2 \\ 1/2 & -\hat{\lambda} \end{vmatrix} = -\hat{\lambda}\left(-\frac{3}{4} - \hat{\lambda}\right) - \frac{1}{2}\cdot\frac{1}{2} = \hat{\lambda}^2 + \frac{3}{4}\hat{\lambda} + \frac{1}{4}$$

$$\hat{\lambda}^2 + \frac{3}{4}\lambda + \frac{1}{4} = 0 = (\hat{\lambda}+1)(\hat{\lambda} - \frac{1}{4})$$

$$\hat{\lambda}_1 = -1 \quad \hat{\lambda}_2 = \frac{1}{4}$$

d. *(2 points).* Confirm that the eigenvectors of A^{-1} are the same eigenvectors, $\vec{x}_1$ and $\vec{x}_2$, as A (from part b.)

$E_{-1} = \text{null}(A+I) = \text{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$ $\qquad E_{1/4} = \text{null}(A - \frac{1}{4}I) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$

$+I = \left(\begin{array}{cc|c} 1/4 & 1/2 & 0 \\ 1/2 & 1 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\qquad A - \frac{1}{4}I = \left(\begin{array}{cc|c} -1 & 1/2 & 0 \\ 1/2 & -1/4 & 0 \end{array}\right)$

$\qquad 6\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$