Math 214 Fall 2003 **SHOW ALL YOUR WORK** Quiz Six

1. Consider the matrix $A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$. We want to obtain a value for $A^\infty = \lim_{n\to\infty} A^n$.

a. (*4 points*). Find the eigenvalues and eigenvectors of A.

$$p(\lambda) = \begin{vmatrix} 1/2-\lambda & 1/2 \\ 1 & -\lambda \end{vmatrix} = -\lambda\left(\tfrac{1}{2}-\lambda\right) - \tfrac{1}{2} = \lambda^2 - \tfrac{1}{2}\lambda - \tfrac{1}{2} = 0$$

$$(\lambda-1)\left(\lambda+\tfrac{1}{2}\right) = 0 \Rightarrow \lambda_1 = 1 \text{ or } \lambda_2 = -\tfrac{1}{2}$$

$$\lambda_1 = 1 \quad \left(\begin{array}{cc|c} -1/2 & 1/2 & 0 \\ 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad -x_1 + x_2 = 0, \; x_2 \text{ any}, \; \vec{x}_1 = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1/2 \quad \left(\begin{array}{cc|c} 1 & 1/2 & 0 \\ 1 & 1/2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad 2x_1 + x_2 = 0, \; x_2 \text{ any}, \; \vec{x}_2 = \begin{pmatrix} -\frac{x_2}{2} \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

b. (*2 points*) Show that $AS = S\Lambda$ or $A = S\Lambda S^{-1}$, where the columns of S are formed by the eigenvectors of A and Λ is a diagonal matrix with the eigenvalues of A along the diagonal.

$$S = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \qquad AS = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1 & -1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix} \qquad S\Lambda = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1 & -1 \end{pmatrix}$$

$$S^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

c. (*2 points*). Compute $A^n = S\Lambda^n S^{-1}$.

$$A^n = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1^n & 0 \\ 0 & (-1/2)^n \end{pmatrix}\begin{pmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{pmatrix} = \begin{pmatrix} 1^n & -\left(\frac{1}{2}\right)^n \\ 1^n & 2\left(\frac{1}{2}\right)^n \end{pmatrix}\begin{pmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{pmatrix}$$

$$A^n = \begin{matrix} 2/3\,1^n + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^n & 1/3 \cdot 1^n - \left(\frac{1}{2}\right)^n \cdot \frac{1}{3} \\ 2/3 \cdot 1^n - 2/3\left(\frac{1}{2}\right)^n & \frac{1}{3}\,1^n + \frac{2}{3}\left(\frac{1}{2}\right)^n \end{matrix}$$

d. (*2 points*). Use your answer from **c** to show that $A^\infty = \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}$.

$$A^\infty = \lim_{n\to\infty} A^n = \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix}$$