1. TRUE or FALSE – put your answer in the box. That answer is worth 1 point. To receive ANY credit, you must also give a brief, and correct, explanation in support of your answer! Remember a statement is TRUE only if it is ALWAYS true, and it is FALSE if there exists an example which makes it FALSE.

(a) (3 points) For any $n \times n$ matrix $A$ there exists a real number $\lambda$ and a $n \times 1$ vector $\vec{v}$ such that $A\vec{v} = \lambda \vec{v}$.

FALSE/TRUE

A real matrix can have eigenvalues that are not real numbers.

FALSE

Counterexample: \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

\[
p(A) = \begin{vmatrix}
-\lambda & 1 \\
1 & -\lambda
\end{vmatrix} = \lambda^2 + 1 = 0
\]

\[
\lambda = \pm i
\]

TRUE If $\vec{v} = \vec{0}$ then $A\vec{v} = \lambda \vec{v}$ for all matrices $A$ and all real numbers $\lambda$. So, technically the statement is true if one includes the zero vector as a possibility.

(b) (3 points) $A$ is singular (not invertible) if and only if $A$ has at least one zero eigenvalue. In other words, IF $A$ is singular, THEN $A$ has at least one zero eigenvalue AND IF $A$ has at least one zero eigenvalue, THEN $A$ is singular.

TRUE

$A^{-1}$ exists $\iff$ $\text{det}(A) \neq 0$ $\iff$ $\prod_{i=1}^{n} \lambda_i \neq 0$ $\iff$ $\forall \lambda_i$ $\neq 0$

$A^{-1}$ exists $\iff$ Every $\lambda_i \neq 0$

$A^{-1}$ does not exist $\iff$ There exist a $\lambda_i = 0$

Or you can prove each direction separately:

$A$ singular $\Rightarrow$ $\det(A) = 0$ $\Rightarrow$ $\det(A - \lambda I) = 0$ $\Rightarrow$ $p(\lambda) = 0$ $\Rightarrow$ $\lambda$ is an eigenvalue of $A$.

$\lambda = 0$ $\Rightarrow$ $\sum_{i=1}^{n} \lambda_i = 0$ $\Rightarrow$ $\forall \lambda_i = \det(A) = 0$ $\Rightarrow$ $A$ is singular.

(c) (4 points) The eigenvectors of $A^T$ are the same as the eigenvectors of $A$.

FALSE

Counterexample

$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ $\lambda(A) = 1, 2$

$\lambda(A^T) = 1, 2$

$\text{null}(A - 1I) = \text{null}(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}) = \text{span}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = E_1(A)$

$\text{null}(A^T - 1I) = \text{null}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) = \text{span}(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = E_1(A^T)$

$\text{null}(A - 2I) = \text{null}(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = \text{span}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = E_2(A)$

$\text{null}(A^T - 2I) = \text{null}(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}) = \text{span}(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = E_2(A^T)$

The eigen values of $A^T$ and $A$ are the same but the eigenvectors are different (unless $A = A^T$).