1. 6 points. Consider the linear system
\[
\begin{align*}
4x - 2y + z &= a \\
x + y + z &= b
\end{align*}
\]
where \(a\) and \(b\) are real numbers. Our goal is to discover a relationship between the solution sets of this system for various values of \(a\) and \(b\).

a. 2 points. Consider the case \(a = b = 0\). This is known as the homogeneous case. Use Gaussian Elimination to solve the system.

\[
\begin{align*}
\begin{pmatrix} 4 & -2 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 4 & -2 & 1 \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -6 & -3 \\ 0 & 2 & 1 \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1/2 \\ 0 & 2 & 1 \end{pmatrix}
\end{align*}
\]

\[R_2' = R_2 + 4R_1, \quad R_3' = R_3 - 2R_1\]

\[
\begin{align*}
\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1/2 \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1/2 \\ 0 & 2 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x + \frac{1}{2}z = 0 \\ 2x + z = 0 \end{pmatrix} \\
\frac{x - z}{2} &= -z \quad z = t \quad x = -\frac{1}{2}z = -\frac{1}{2}t
\end{align*}
\]

b. 2 points. What is the geometric interpretation or “shape” of the solution? Is it a point in \(\mathbb{R}^2\)? A point in \(\mathbb{R}^3\)? A line in \(\mathbb{R}^2\)? A line in \(\mathbb{R}^3\)? A plane in \(\mathbb{R}^3\)? Something else?

The solution is a line through the origin in \(\mathbb{R}^3\).

c. 2 points. Express your solution in vector form, i.e. \(\overrightarrow{x} = \overrightarrow{p} + t\overrightarrow{d}\).

\[
\overrightarrow{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}
\]

2. 4 points. Choose any non-zero value of \(a\) and \(b\) that you like. This is known as the non-homogeneous case.

a. 2 points. Repeat Question 1 (i.e. Use Gaussian Elimination to solve the system with your chosen values of \(a\) and \(b\)) and express your answers in vector form, i.e. \(\overrightarrow{x} = \overrightarrow{p} + t\overrightarrow{d}\).

\[
\begin{align*}
\begin{pmatrix} 1 & 1 & 1 \\ 4 & -2 & 1 \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -6 & -3 \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix} x + \frac{1}{2}z = \frac{1}{2} \\ y + \frac{1}{2}z = \frac{1}{2} \end{pmatrix} \Rightarrow \begin{pmatrix} x = \frac{1}{2} - \frac{1}{2}z \\ y = \frac{1}{2} - \frac{1}{2}z \end{pmatrix} \Rightarrow \overrightarrow{x} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}
\end{align*}
\]

b. 2 points. What is the (geometric) relationship between your solutions in 1(c) and 2(a)? In other words, how are the solutions to the homogeneous linear system and non-homogeneous linear system related? EXPLAIN YOUR ANSWER.

The solution to the non-homogeneous system is a line parallel (same direction vector) but shifted so that it does not go through the origin, as the solution to the homogeneous system.