# Linear Systems





*Class 7*: **Monday February 6**

**SUMMARY** Reduced Row Echelon Form and Rank
**CURRENT READING** Poole 2.2

**Summary**
We will discuss the reduced row echelon form of a matrix, sometimes denoted `rref(A)`, and introduce the important concept of rank of a matrix.

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**Homework Assignment**
*HW #7: Section 2.2: 1, 2, 3, 4, 5, 6, 7, 8,* **22, 23,26**, *27, 28, 29, 36,* **43**. *EXTRA CREDIT # 47.*

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## 1. Reduced Row Echelon Form

Definition

A matrix is in **reduced row echelon form** if it satisfies the following property:

1. It is in row echelon form (i.e. all zero rows are at the bottom and the first non-zero entry of each non-zero row is in a column to the left of any non-zero leading entry in rows below it, forming an echelon or staircase appearance).
2. The leading entry is in each non-zero row is 1.
3. Each column containing a leading entry of 1 has zeos everwhere else.

GroupWork

Find the reduced row echelon matrix for each of the following. (Also note what are the dimensions (i.e. number of rows and number of columns) for each matrix?)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 2 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 0 & 4 & 8 \end{bmatrix}$$

## 2. Definition of Rank

*Definition* 1. The rank of a $m \times n$ matrix $A$ is the number of non-zero rows in its row echelon form. We call this number $r = \text{rank(A)}$.

By definition $r \leq m$ and $r \leq n$. Also, $r$ is the number of pivots a coefficient matrix has when applying Gaussian Elimination.

**THEOREM**

When there are more columns than rows in a matrix, i.e. $n > m$ there is ALWAYS atleast one free variable. In other words if $A$ is the $m \times n$ coefficient matrix of a consistent linear system, the number of free variables $= n - r$. This result is known as **the Rank Theorem.**

**Exercise**

What are the rank of each of the matrices $A$, $B$ and $C$ given on the other side of this page? How is $r$ related to $m$ and $n$ in each case? Do you notice a pattern?

EXAMPLE

Let's solve the following linear system of equations

$$\begin{aligned} x + 2y - z &= 3 \\ 2x + 3y + z &= 1 \end{aligned}$$

EXAMPLE

Consider $\vec{p} = (1, 0, -1)$, $\vec{q} = (0, 2, 1)$, $\vec{u} = (1, 1, 1)$ and $\vec{v} = (3, -1, -1)$ Do the following lines $\vec{x} = \vec{p} + t\vec{q}$ and $\vec{x} = \vec{u} + t\vec{v}$ intersect? If so, where? Is it possible for lines in $\mathbb{R}^3$ to not intersect and not be parallel?