Multivariable Calculus

Math 212 Spring 2015 Fowler 309 MWF 9:35am - 10:30am
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Worksheet 28

TITLE The Three Fundamental Theorems of Multivariable Calculus
CURRENT READING McCallum, Section 20.3)
HW #12 (DUE TUESDAY 04/28/15 5PM)
McCallum, *Section 18.4*: 1, 2, 3, 4, 15, 16, 20, 23*.
McCallum, *Chapter 18 Review*: 1, 2, 8, 15, 16, 17, 26, 45.
McCallum, *Section 19.3*: 1, 2, 3, 4, 6, 11, 27, 28.
McCallum, *Section 20.1*: 3, 4, 7, 13, 14, 28.

SUMMARY
This worksheet discusses the three fundamental theorems of Multivariable (Vector) Calculus: Fundamental Theorem of Line Integrals, Stokes' Theorem and the Divergence Theorem (sometimes called Gauss' Theorem).

RECALL

Given a vector field $\vec{F}$ in $\mathbb{R}^3$ such that $\vec{F} = F_1(x,y)\hat{\imath} + F_2(x,y)\hat{\jmath} + 0\hat{k}$ the expression

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = (\vec{\nabla} \times \vec{F}) \cdot \hat{k}$$

is sometimes called the "scalar curl."

That expression for the scalar curl appears in Green's Theorem, as the integrand of the double integral on the lefthand side (LHS):

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy = \oint_\gamma F_1\, dx + F_2\, dy \qquad \textbf{(Green's Theorem)}$$

If we put these two thoughts together we get...

THEOREM: Stokes' Theorem in the Plane

So, in vector format, we can write Green's Theorem as

$$\iint_D (\vec{\nabla} \times \vec{F}) \cdot \hat{k}\, dA = \oint_{\partial D} \vec{F} \cdot d\vec{x} \qquad \textbf{(Stokes' Theorem in the Plane)}$$

When written this way, Green's Theorem is also known as Stokes' Theorem in the Plane.

NOTE: The symbol ∂D means the boundary of the region D. So, Stokes' Theorem in the Plane says that the line integral of a vector field around a closed path ∂D is equal to the area integral of the curl through the region D bounded by the path ∂D.

Gauss Theorem: An Application of Green's Theorem

Let's look closely at the expression $\vec{F} \cdot d\vec{x}$ in the line integral RHS of Green's Theorem. If we consider a unit tangent vector $\hat{t}$ that points in the direction of travel along the path ∂D then we can write $d\vec{x} = ds\,\hat{t}$ where ds is an infinitesimal section of the curved path and s is parameter corresponding to the length of the curve. (**See Figure 1**)

Suppose we had a unit vector $\hat{n}$ which is orthogonal in the plane (i.e. normal) to the boundary curve ∂D at every point. This means that $\hat{n}$ is at 90 degrees (orthogonal) to the tangent vector $\hat{t}$ and points **away** from the interior of a closed region D where ∂D is a path which makes up the boundary of D. Mathematically, then $\hat{n} \cdot \hat{t} = 0$.

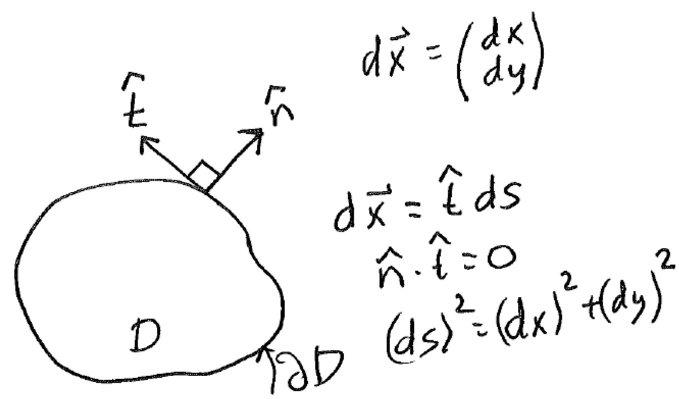

Figure 1: Shows relationship of $\hat{n}$ and $\hat{t}$

In Stokes' Theorem in the Plane (a.k.a Green's Theorem) we are dealing with a vector field in $\mathbb{R}^3$ which has the form $\vec{F} = F_1(x,y)\hat{i} + F_2(x,y)\hat{j} + 0\hat{k}$.

Let's consider what would happen if we considered a vector field orthogonal to $\vec{F}$ defined to be $\vec{H} = -F_2(x,y)\hat{i} + F_1(x,y)\hat{j} + 0\hat{k}$. **Gauss' Theorem** is about trying to evaluate the line integral along a boundary curve for this new orthogonal field. (Note that $\vec{H} \cdot \vec{F} = 0$.)

We can show that $\vec{H} \cdot d\vec{x} = \vec{F} \cdot \hat{n}\, ds$.

$$\begin{aligned}
\vec{H} \cdot d\vec{x} &= [-F_2(x,y)\hat{i} + F_1(x,y)\hat{j} + 0\hat{k}] \cdot [dx\hat{i} + dy\hat{j} + 0\hat{k}] \\
&= -F_2(x,y)dx + F_1(x,y)dy \\
&= [F_1(x,y)\hat{i} + F_2(x,y)\hat{j} + 0\hat{k}] \cdot [dy\hat{i} - dx\hat{j} + 0\hat{k}] \\
\vec{H} \cdot d\vec{x} &= \vec{F} \cdot \hat{n}\, ds
\end{aligned}$$

(Please note that $d\vec{x} = dx\hat{i} + dy\hat{j}$ and $ds\,\hat{n} = -dy\hat{i} + dx\hat{j}$ so that $(ds)^2 = (dx)^2 + (dy)^2 = ||d\vec{x}||^2$.

Now, what what happens if we apply Green's Theorem using $\vec{H}$

$$\begin{aligned}
\int_{\partial D} \vec{H} \cdot d\vec{x} &= \iint_D (\vec{\nabla} \times \vec{H}) \cdot \hat{k}\, dA \\
\int_{\partial D} -F_2(x,y)\, dx + F_1(x,y)\, dy &= \iint_D \left(\frac{\partial}{\partial x}[F_1(x,y)] - \frac{\partial}{\partial y}[-F_2(x,y)] \right) dA \\
\int_{\partial D} \vec{F} \cdot \hat{n}\, ds &= \iint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) dA \\
&= \iint_D \vec{\nabla} \cdot \vec{F}\, dA
\end{aligned}$$

By noticing that the right hand side of this expression $\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$ is exactly the divergence of $\vec{F}$, i.e. $\vec{\nabla} \cdot \vec{F}$ this application of Green's theorem produces the result that a line integral of the *orthogonal complement* of a vector field around a boundary of a planar region is equal to the area integral of the divergence of that vector field over that same region.

THEOREM: Gauss' Theorem in the Plane

$$\int_{\partial D} \vec{F} \cdot \hat{n}\, ds = \iint_D \vec{\nabla} \cdot \vec{F}\, dA$$

The Fundamental Theorems of Calculus

THEOREM: Divergence Theorem in $\mathbb{R}^3$

$$\iiint_D \vec{\nabla} \cdot \vec{F}\, dV = \iint_{\partial D} \vec{F} \cdot d\vec{A} \qquad \textbf{(Gauss' Theorem in Space)}$$

The Divergence Theorem is the most general form or the Gauss' Theorem, equating the integral of the divergence of a vector field $\vec{F}(\vec{x})$ through a volume of space D to the surface area integral of $\vec{F}$ over the boundary of the region, called ∂D.

THEOREM: Stokes' Theorem in $\mathbb{R}^3$

Stokes' Theorem (Sometimes called the Curl Theorem) in space equates the surface area integral of the curl of a vector field $\vec{F}$ over a particular surface D with the line integral of that same vector field $\vec{F}$ over the boundary of that surface, i.e. ∂D.

$$\iint_D (\vec{\nabla} \times \vec{F}) \cdot\ d\vec{A} = \oint_{\partial D} \vec{F} \cdot d\vec{x} \qquad \textbf{(Stokes' Theorem in Space)}$$

RECALL

The **Fundamental Theorem of Calculus for Line Integrals**

The Fundamental Theorem of Line Integrals equates the line integral of the gradient of a function to the change in the function between the two end points of the path (i.e. the boundary of the curve).

$$\int_C \vec{\nabla} f \cdot\ d\vec{x} = f(\vec{x}_{end}) - f(\vec{x}_{start}) \qquad \textbf{(Fundamental Theorem of Line Integrals)}$$

$$\int_a^b f'(x)\, dx = f(b) - f(a) \qquad \textbf{(Fundamental Theorem of Calculus)}$$

CONCEPTUAL UNDERSTANDING OF THE FUNDAMENTAL THEOREMS

All these theorems are basically just more complicated versions of a single idea:

SUM OF A RATE OF CHANGE OF A FUNCTION OVER SOME PART OF IT'S INPUT DOMAIN = CHANGE IN OUTPUT OF THAT FUNCTION FROM START TO END

Fundamental Theorem of Calculus

For the FTC, the rate of change is simply the **derivative** of f and the input domain is a range of its input domain. The FTC says this is equal to the change in the function along this range of the domain.

Fundamental Theorem of Line Integrals

For the FTLI, the rate of change is the **gradient** of f and the input domain is a path in space. The FTLI says this is equal to the change in the function f at the beginning and end of the domain.

Stokes' Theorem a.k.a. the Curl Theorem

For Stokes' Theorem, the rate of change is the **curl** of a vector field and the input domain is an oriented surface. Stokes' theorem says this is equal to the circulation of this vector field (i.e. line integral) around the boundary of the surface (i.e. a curved path in space).

Gauss' Theorem a.k.a. the Divergence Theorem

For Gauss' Theorem, the rate of change is the **div** of a vector field and the input domain is a volume of space. Gauss' theorem says this is equal to the flux of this vector field (i.e. surface area integral) around the boundary of the volume (i.e. a surface).