Worksheet 27

TITLE Divergence and Curl of a Vector Field

CURRENT READING McCallum, Section 19.3 and 20.1

HW #12 (DUE TUESDAY 04/28/15 5PM)

McCallum, Section 18.4: 1, 2, 3, 4, 15, 16, 20, 23*.
McCallum, Chapter 18 Review: 1, 2, 8, 15, 16, 17, 26, 45.
McCallum, Section 19.3: 1, 2, 3, 4, 6, 11, 27, 28.
McCallum, Section 20.1: 3, 4, 7, 13, 14, 28.

SUMMARY
This worksheet discusses the geometric and algebraic definitions of the curl and divergence of a vector field.

RECALL
Given a vector field $\vec{F}$ in $\mathbb{R}^2$ such that $\vec{F} = F_1(x, y)\hat{i} + F_2(x, y)\hat{j}$ the expression $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ is called the scalar curl.

DEFINITION: scalar curl in $\mathbb{R}^3$
Given a 3-D vector field with only two components $\vec{F}(x, y) = F_1(x, y)\hat{i} + F_2(x, y)\hat{j} + 0\hat{k}$ we can define the (badly-misnamed) scalar curl of $\vec{F}$ to be

$$\text{curl } \vec{F} = \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k}$$

NOTE: The curl of a 2-D vector field will either be pointed into the page (using the symbol $\otimes$) or out of the page (using the symbol $\odot$) or be the zero vector $\vec{0}$.

The Curl Of A Vector Field

DEFINITION: vector curl in $\mathbb{R}^3$
The curl of a vector field $\vec{F}(x) \in \mathbb{R}^3$ is a vector property denoted by $\text{curl } \vec{F}(x)$ and defined as $\vec{\nabla} \times \vec{F}$ where $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ in $\mathbb{R}^3$ and $\vec{\nabla}$ is the vector operator $\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$.

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{i} + \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \hat{j} + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k}$$

Algebraically, you could think of the curl as the following determinant:

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

NOTE: The curl is the cross product of the gradient operator with a vector field $\vec{F}$, so it is a vector quantity.
The Curl Of Some Of Our Favorite Vector Fields

**EXAMPLE**

Let’s find the curl of some of our favorite planar vector fields.

\[ \vec{A}(x, y) = (x + y)\hat{i} + (x - y)\hat{j} \]

\[ \vec{B}(x, y) = x\hat{i} + y\hat{j} \]

\[ \vec{C}(x, y) = x\hat{j} \]

\[ \vec{D}(x, y) = -y\hat{i} \]

\[ \vec{E}(x, y) = y\hat{i} - x\hat{j} \]

\[ \vec{F}(x, y) = -y\hat{i} + x\hat{j} \]
**Geometric Understanding Of Curl**

**DEFINITION: circulation density**

The **circulation density** of a smooth vector field $\vec{F}$ around the direction of a unit vector $\hat{n}$ is defined, provided the limit exists, to be

$$\text{circ}_{\hat{n}} \vec{F} = \lim_{\text{Area} \to 0} \frac{\oint_{C} \vec{F} \cdot d\vec{x}}{\text{Area inside } C} = \lim_{\text{Area} \to 0} \frac{\text{Circulation of } \vec{F} \text{ around } C}{\text{Area inside } C} = (\vec{\nabla} \times \vec{F}) \cdot \hat{n}$$

where $C$ is a closed curve in a plane perpendicular to $\hat{n}$ positively oriented using the right-hand rule. (When the Right-Hand Thumb points in direction of $\hat{n}$ Other Fingers are curled in direction of traversal around $C$.)

**CONCEPTUAL UNDERSTANDING OF CURL**

The direction the vector **curl** of a vector field $\vec{F}$ points in is the direction for which the circulation density of $\vec{F}$ is the GREATEST.

The magnitude of the vector **curl** of a vector field $\vec{F}$ is the circulation density of $\vec{F}$ around the direction $\vec{\nabla} \times \vec{F}$ points in.

If the circulation density is zero around every direction then we say the curl is $\vec{0}$ and describe such a vector field as **irrotational**.

Recall that gradient fields have the property that every line integral around a closed path is zero so this means that all gradient fields are irrotational, which can be expressed mathematically as

$$\vec{\nabla} \times \vec{\nabla} \phi = \vec{0} \text{ for any potential function } \phi$$

**Divergence of a Vector Field**

**DEFINITION: divergence**

The **divergence** of a vector field $\vec{F}(\vec{x})$ is a **scalar** property denoted by $\text{div} \vec{F}(\vec{x})$ defined as the trace of the Jacobian matrix, i.e. the sum of the diagonal elements of this matrix. In particular, if one considers $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$ in $\mathbb{R}^3$ where $\vec{\nabla} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$ then the divergence of $\vec{F}$ can be defined as

$$\text{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

**NOTE:** The divergence is the **dot product** of the gradient operator with a vector field $\vec{F}$, so it is a scalar quantity.

**GROUP WORK**

Find the Divergence of the six vector fields depicted earlier on this worksheet.
**Properties Of Gradient, Divergence and Curl As Vector Calculus Operations**

The divergence $\nabla \cdot \mathbf{F}$ and curl $\nabla \times \mathbf{F}$ can be thought of as differential operations that are applied to vector fields, and produce scalars and vectors, respectively. The gradient operator $\nabla$ is applied to a scalar function and outputs a vector field. Given scalar functions $\phi$ and $\psi$ and vector fields $\mathbf{F}$ and $\mathbf{G}$ the following properties apply to the gradient, curl and divergence operators.

**Distributivity**

\[
\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}
\]
\[
\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}
\]
\[
\nabla(\phi + \psi) = \nabla \phi + \nabla \psi
\]

**Product Rules**

\[
\nabla \cdot (\phi \mathbf{F}) = \mathbf{F} \cdot (\nabla \phi) + \phi \nabla \cdot \mathbf{F}
\]
\[
\nabla \times (\phi \mathbf{F}) = \phi (\nabla \times \mathbf{F}) + (\nabla \phi) \times \mathbf{F}
\]
\[
\nabla(\phi \psi) = \psi (\nabla \phi) + \phi (\nabla \psi)
\]

**Repeated Applications** (“Second Derivatives”)

\[
\nabla \cdot (\nabla \times \mathbf{F}) = \text{div curl } \mathbf{F} = 0
\]
\[
\nabla \times (\nabla \phi) = \text{curl grad } \phi = 0
\]
\[
\nabla \cdot (\nabla \phi) = \nabla^2 \phi = \Delta \phi = \text{div grad } \phi
\]
\[
\nabla \times (\nabla \times \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} = \text{curl curl } \mathbf{F}
\]

**Exercise**

How many possible binary arrangements of $\text{div}$, $\text{grad}$ and $\text{curl}$ are there?

**QUESTION:** How many of these are well-defined operations? How many of these are identically zero?

**EXAMPLE**

For $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$ and $\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}$ let’s confirm the vector calculus identities

\[
\text{div curl } \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0 \quad \text{and} \quad \nabla \times (\nabla \phi) = \text{curl grad } \phi = 0.
\]