Multivariable Calculus

Math 212 Spring 2015 Fowler 309 MWF 9:35am - 10:30am
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Worksheet 25

TITLE Gradient Fields and Path Independence
CURRENT READING McCallum, Section 18.3)
HW #11 (DUE Wednesday 04/22 BY 5PM)
McCallum, *Section 18.1*: 6, 11,12,13,14, 22, 27.
McCallum, *Section 18.2*: 4, 5, 6, 7, 8, 20, 33..
McCallum, *Section 18.3*: 3, 4, 5, 6, 18, 21, 30.

SUMMARY
This worksheet discusses when one can use then Fundamental Theorem of Line Integrals to more simply evaluate line integrals regardless of the path taken.

RECALL

The Fundamental Theorem of Calculus says

$$\int_a^b f'(t)dt = f(b) - f(a)$$

There is a corresponding principle for line integrals, called the Fundamental Theorem of Line Integrals.

The Fundamental Theorem of Line Integrals

THEOREM

Given that C is a piecewise smooth oriented path which starts at $\vec{x}_A$ and ends at $\vec{x}_B$. If f is a function whose gradient $\vec{\nabla} f$ is continuous on the path C, then

$$\int_C \vec{\nabla} f \cdot d\vec{x} = f(\vec{x}_B) - f(\vec{x}_A)$$

CONCEPTUAL UNDERSTANDING

The Fundamental Theorem of Line Integrals means that regardless of the path taken from $\vec{x}_A$ to $\vec{x}_B$ the value of a line integral in a gradient field is the same, in other words **line integrals of gradient fields are path-independent**. This is fantastic because it means we can evaluate line integrals without having to worry about parametrizations of paths whatsoever. In other words, **ALL GRADIENT FIELDS ARE PATH-INDEPENDENT.**

DEFINITION: **conservative or path-independent vector field**

A vector field $\vec{F}$ is said to be **conservative** or **path-independent**, if for any two points $\vec{x}_A$ and $\vec{x}_B$, the line integral $\int_C \vec{F} \cdot d\vec{r}$ has the same value along ANY piecewise smooth path C lying in the domain of $\vec{F}$ that connects the points $\vec{x}_A$ and $\vec{x}_B$.

EXAMPLE

Given that the vector field $\vec{F}(x,y) = \vec{\nabla} f$ where $f(x,y) = \dfrac{x^2}{2} + \dfrac{y^2}{2}$ find the value of $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ where C is the circular arc in the counterclockwise direction from $(2,0)$ to $(0,2)$.

THEOREM

If a vector field $\vec{F}$ is a continuous and path-independent on an open region $\mathcal{R}$, then there exists an f defined on $\mathcal{R}$ so that **grad** f equals $\vec{F}$.

DEFINITION: **potential function**

This function f which can be used to generate a path-indendent field (i.e. a gradient field) is called the **potential function** for the vector field $\vec{F}$. Physicists and applied mathematicians like to use the symbol ϕ for the potential, so that $\vec{F} = \vec{\nabla}\phi$.

CONCEPTUAL UNDERSTANDING

The theorem says that all path-independent fields $\vec{F}$ must possess a potential function f which can be used to generate the field $\vec{F}$ by taking the gradient of f. But this means that path-independent fields are gradient fields.

ALL PATH-INDEPENDENT VECTOR FIELDS ARE GRADIENT FIELDS

So, one way to show that a give vector field is path-independent is to find a potential function for that field.

Exercise

McCallum, page 978, Example 3. Show that the vector field $\vec{F}(x,y) = y\cos x\hat{i} + (\sin x + y)\hat{j}$ is path-independent.