Worksheet 20

TITLE Triple Integrals
CURRENT READING McCallum, Section 16.3
HW #8 (DUE Wednesday 04/01/15 5PM)
McCallum, *Section 16.1*: 2, 4, 6, 7, 8, 14, 22, 23..
McCallum, *Chapter 16.2*: 1, 3, 4, 7, 11, 13, 14, 16, 18, 19, 23, 33, 34, 37, 38, 43, 50.

SUMMARY
This worksheet discusses ultimate iterated integral; the triple integral! We shall learn the importance of being able to describe a volume in $\mathbb{R}^3$ multiple ways.

`QUESTION:` How would one evaluate the integral $\int_0^x \int_0^1 y^2 x \, dx \, dy$?

The Triple Integral

Consider the following integral

$$\int_p^q \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz$$

You evaluate this integral from the "inside" (the dx integral) out (the dz integral), treating the other variables as constants. So, for the dx integral z and y are constants and for the dy integral the z is a constant and the last integral (leftmost) must have only constants as limits. Always remember that a definite integral, even a triple integral represents a **number**.

NOTE

- The limits for the outer integral can only involve constants.
- The limits for the middle integral can involve only one variable (the variable that is in the outer limit)
- The limits for the inner integral can involve two variables (those from the two outer integrals)
- There are six different possible arrangements of the variables $dx\,dy\,dz$, $dx\,dz\,dy$, $dz\,dy\,dx$, $dz\,dx\,dy$, $dy\,dx\,dz$, and $dy\,dz\,dx$. Fubini's Theorem applies to all 6!

EXAMPLE

Evaluate $\int_0^1 \int_2^4 \int_{-2}^1 xyz \, dx \, dy \, dz$

THEOREM

Fubini's Theorem for Triple Integrals

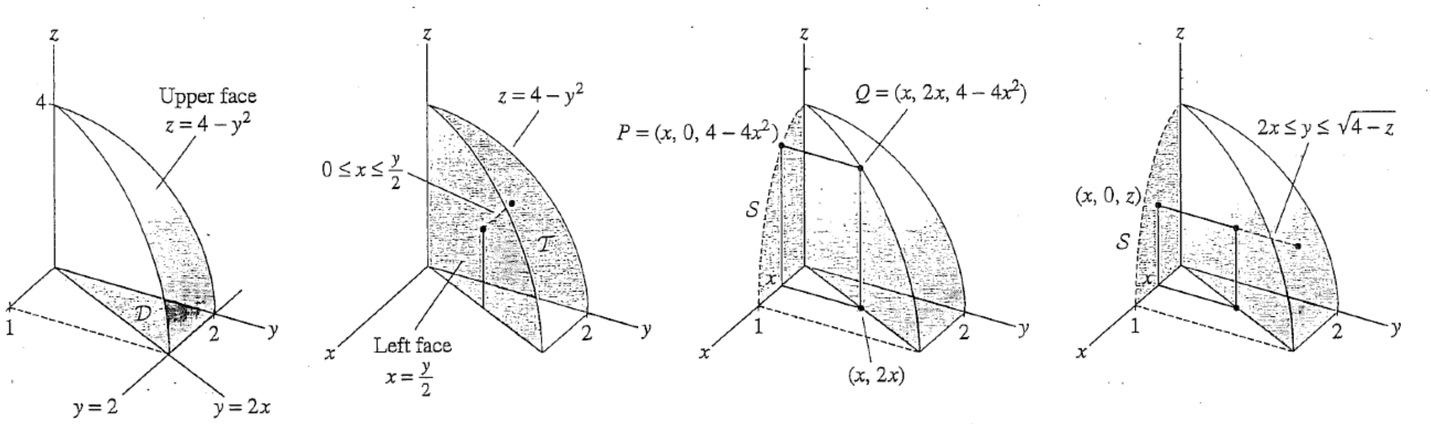
The integral of a continuous function $f(x,y,z)$ over the rectangular volume $\mathcal{R} = [a,b] \times [c,d] \times [p,q]$ is equal to the iterated integral (computed in any order).

$$\begin{aligned}
\int_p^q \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz &= \int_p^q \int_a^b \int_c^d f(x,y,z)\,dy\,dx\,dz = \int_a^b \int_p^q \int_c^d f(x,y,z)\,dy\,dz\,dx \\
&= \int_a^b \int_c^d \int_p^q f(x,y,z)\,dz\,dy\,dx = \int_c^d \int_a^b \int_p^q f(x,y,z)\,dz\,dx\,dy \\
&= \int_c^d \int_p^q \int_a^b f(x,y,z)\,dx\,dz\,dy
\end{aligned}$$

GROUPWORK

McCallum, page 832, Example 3.

Evaluate the integral $\int_0^1 \int_2^4 \int_{-2}^1 xyz\,dx\,dy\,dz$ in at least two different ways and show that you get the same answer regardless of order of integration. (Thank Fubini!)



A) Projection to xy-plane (B) Projection to yz-plane (C) Projection to xz-plane

Integration Of A Function Over A Solid Region With Non-Rectangular Base

Let $\mathcal{W}$ be the region in $\mathbb{R}^3$ bounded by the curves $z = 4 - y^2$, $y = 2x$ and $z = 0$ and $x = 0$. We can re-write the triple integral representing the volume of this region **three** different ways, by projecting on to the xy-plane, or the xz-plane or the yz-plane.

METHOD 1: xy-plane
The projection of the region $\mathcal{W}$ in the xy-plane means that $z = 0$, so it looks like $0 \leq x \leq 1$ and $2x \leq y \leq 2$. With the xy-plane as a base, z varies from $z = 0$ to $z = 4 - y^2$.

$$\text{Volume} = \int_{x=0}^{1}\int_{y=2x}^{2}\int_{z=0}^{4-y^2} 1\; dz\; dy\; dx$$

METHOD 2: yz-plane
The projection of the region $\mathcal{W}$ in the yz-plane means that $x = 0$, so it looks like $0 \leq y \leq 1$ and $0 \leq z \leq 4 - y^2$. With the yz-plane as a base, x varies from $x = 0$ to $x = y/2$, since we knew that $y = 2x$ was a boundary of $\mathcal{W}$.

$$\text{Volume} = \int_{y=0}^{2}\int_{z=0}^{4-y^2}\int_{x=0}^{y/2} 1\; dx\; dz\; dy$$

METHOD 3: The xz-plane
The projection of the region $\mathcal{W}$ in the xz-plane means that $y = 0$. But it also means we have to figure what happens to the $z = 4 - y^2$ curve when it gets projected onto the xz-plane. Since we know $y = 2x$, this means that $z = 4 - (2x)^2 = 4 - 4x^2$, so it looks like in the xz-plane $0 \leq x \leq 1$ and $0 \leq z \leq 4 - 4x^2$. With the xz-plane as a base, we need to figure out how y varies with x and z. But since $z = 4 - y^2$ that means that $y = \pm\sqrt{4 - z}$ which since we are looking at the region above the xz-plane means we take the positive square root.

$$\text{Volume} = \int_{x=0}^{1}\int_{z=0}^{4-4x^2}\int_{y=2x}^{\sqrt{4-z}} 1\; dy\; dz\; dx$$

EXAMPLE

Let's compute the first of these integrals to find the volume of the given shape.

$$\int_{y=0}^{2}\int_{z=0}^{4-y^2}\int_{x=0}^{y/2} 1\ dx\ dz\ dy$$

Exercise

You should evaluate at least one of the other integrals to show that you get the same answer.

$$\int_{x=0}^{1}\int_{z=0}^{4-4x^2}\int_{y=2x}^{\sqrt{4-z}} 1\ dy\ dz\ dx$$

$$\int_{x=0}^{1}\int_{y=2x}^{2}\int_{z=0}^{4-y^2} 1\ dz\ dy\ dx$$