Worksheet 20

TITLE Triple Integrals
CURRENT READING McCallum, Section 16.3
HW #8 (DUE Wednesday 04/01/15 5PM)
McCallum, Section 16.1: 2, 4, 6, 7, 8, 14, 22, 23..  
McCallum, Chapter 16.2: 1, 3, 4, 7, 11, 13, 14, 16, 18, 19, 23, 33, 34, 37, 38, 43, 50.

SUMMARY
This worksheet discusses ultimate iterated integral; the triple integral! We shall learn the importance of being able to describe a volume in $\mathbb{R}^3$ multiple ways.

QUESTION: How would one evaluate the integral $\int_0^2 \int_0^1 y^2 x \, dx \, dy$?

The Triple Integral
Consider the following integral
\[
\int_p^q \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz
\]
You evaluate this integral from the “inside” (the $dx$ integral) out (the $dz$ integral), treating the other variables as constants. So, for the $dx$ integral $z$ and $y$ are constants and for the $dy$ integral the $z$ is a constant and the last integral (leftmost) must have only constants as limits. Always remember that a definite integral, even a triple integral represents a number.

NOTE
- The limits for the outer integral can only involve constants.
- The limits for the middle integral can involve only one variable (the variable that is in the outer limit)
- The limits for the inner integral can involve two variables (those from the two outer integrals)
- There are six different possible arrangements of the variables $dx \, dy \, dz$, $dx \, dz \, dy$, $dz \, dy \, dx$, $dz \, dx \, dy$, $dy \, dx \, dz$, and $dy \, dz \, dx$. Fubini’s Theorem applies to all 6!

EXAMPLE
Evaluate $\int_0^1 \int_2^4 \int_{-2}^1 xyz \, dx \, dy \, dz$
**THEOREM**

**Fubini’s Theorem for Triple Integrals**

The integral of a continuous function \( f(x, y, z) \) over the rectangular volume \( \mathcal{R} = [a, b] \times [c, d] \times [p, q] \) is equal to the iterated integral (computed in any order).

\[
\int_p^q \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz = \int_p^q \int_c^d \int_a^b f(x, y, z) \, dy \, dx \, dz = \int_a^b \int_p^q \int_c^d f(x, y, z) \, dy \, dz \, dx = \int_a^b \int_d^c \int_p^q f(x, y, z) \, dz \, dx \, dy = \int_p^q \int_d^c \int_a^b f(x, y, z) \, dx \, dz \, dy
\]

**GROUP WORK**

*McCallum, page 832, Example 3.*

Evaluate the integral \( \int_0^1 \int_2^4 \int_{-2}^1 xyz \, dx \, dy \, dz \) in at least two different ways and show that you get the same answer regardless of order of integration. (Thank Fubini!)
Integration Of A Function Over A Solid Region With Non-Rectangular Base

Let \( W \) be the region in \( \mathbb{R}^3 \) bounded by the curves \( z = 4 - y^2 \), \( y = 2x \) and \( z = 0 \) and \( x = 0 \).

We can re-write the triple integral representing the volume of this region three different ways, by projecting on to the \( xy \)-plane, or the \( xz \)-plane or the \( yz \)-plane.

METHOD 1: \( xy \)-plane
The projection of the region \( W \) in the \( xy \)-plane means that \( z = 0 \), so it looks like \( 0 \leq x \leq 1 \) and \( 2x \leq y \leq 2 \). With the \( xy \)-plane as a base, \( z \) varies from \( z = 0 \) to \( z = 4 - y^2 \).

\[
\text{Volume} = \int_{x=0}^{1} \int_{y=2x}^{2} \int_{z=0}^{4-y^2} 1 \, dz \, dy \, dx
\]

METHOD 2: \( yz \)-plane
The projection of the region \( W \) in the \( yz \)-plane means that \( x = 0 \), so it looks like \( 0 \leq y \leq 1 \) and \( 0 \leq z \leq 4 - y^2 \). With the \( yz \)-plane as a base, \( x \) varies from \( x = 0 \) to \( x = y/2 \), since we knew that \( y = 2x \) was a boundary of \( W \).

\[
\text{Volume} = \int_{y=0}^{2} \int_{z=0}^{4-y^2} \int_{x=0}^{y/2} 1 \, dx \, dz \, dy
\]

METHOD 3: The \( xz \)-plane
The projection of the region \( W \) in the \( xz \)-plane means that \( y = 0 \). But it also means we have to figure what happens to the \( z = 4 - y^2 \) curve when it gets projected onto the \( xz \)-plane. Since we know \( y = 2x \), this means that \( z = 4 - (2x)^2 = 4 - 4x^2 \), so it looks like in the \( xz \)-plane \( 0 \leq x \leq 1 \) and \( 0 \leq z \leq 4 - 4x^2 \). With the \( xz \)-plane as a base, we need to figure out how \( y \) varies with \( x \) and \( z \). But since \( z = 4 - y^2 \) that means that \( y = \pm \sqrt{4 - z} \) which since we are looking at the region above the \( xz \)-plane means we take the positive square root.

\[
\text{Volume} = \int_{x=0}^{1} \int_{z=0}^{4-4x^2} \int_{y=2x}^{\sqrt{4-z}} 1 \, dy \, dz \, dx
\]
EXAMPLE
Let’s compute the first of these integrals to find the volume of the given shape.
\[
\int_2^1 \int_{y=0}^{1-y^2} \int_{z=0}^{y/2} 1 \, dx \, dz \, dy
\]

Exercise
You should evaluate at least one of the other integrals to show that you get the same answer.
\[
\int_1^1 \int_{x=0}^{4-4x^2} \int_{z=0}^{\sqrt{4-z}} 1 \, dy \, dz \, dx
\]
\[
\int_1^1 \int_{y=2x}^{2} \int_{z=0}^{1-y^2} 1 \, dz \, dy \, dx
\]