Title: Integration of a Multivariable Function \( f(x, y) \)

**Current Reading**
McCallum, Section 16.1-16.2

**HW #8 (DUE Wednesday 04/01/15 5PM)**

McCallum, Section 16.1: 2, 4, 6, 7, 8, 14, 22, 23..
McCallum, Chapter 16.2: 1, 3, 4, 7, 11, 13, 14, 16, 18, 19, 23, 33, 34, 37, 38, 43, 50.

**Summary**
This worksheet discusses the concept of the integral of a surface \( f(x, y) \) over a region \( R \), known as a double integral which may be used to compute areas and/or volumes.

**The Definite Integral** \( \int_a^b f(x) \, dx \)

**RECALL**
Given a function \( f(x) \) defined on an interval \( a \leq x \leq b \) the **definite integral** \( \int_a^b f(x) \, dx \) can be defined as

\[
\int_a^b f(x) \, dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{N} f(x_k) \Delta x_k
\]

**CONCEPT**
The definite integral can be approximated by the limit of the Riemann sums and represents the signed area “under” a curve \( y = f(x) \). You can think of this conceptually as a number of rectangles representing the area which in the limit as the width of the rectangles goes to zero (and the number of rectangles goes to infinity) becomes a finite number which is exactly the value of the area.

**EXAMPLE**
Draw a picture representing the right-hand Riemann sum approximations of the integral of the function \( f(x) = x^2 + 1 \) over the interval \( 0 \leq x \leq 3 \).

**Exercise**
What is the average value of the function \( f(x) = x^2 + 1 \) over the interval \( 0 \leq x \leq 3 \)?
The Double Integral \[ \int \int_{\mathcal{R}} f(x, y) \, dA \]
The double integral of \( f(x, y) \) over the rectangular region \( \mathcal{R} \) is defined as

\[
\int \int_{\mathcal{R}} f(x, y) \, dA = \lim_{\max \Delta A_{ij} \to 0} \sum_{i=1}^{M} \sum_{j=1}^{N} f(x_{ij}, y_{ij}) \Delta A_{ij}
\]

OR

\[
\int \int_{\mathcal{R}} f(x, y) \, dA = \lim_{M, N \to \infty} \sum_{i=1}^{M} \sum_{j=1}^{N} f(x_i, y_j) \Delta x_i \Delta y_j
\]

**CONCEPT**

The double integral can be thought of as the three-dimensional analogue to the limit of the Riemann Sums in the single-variable definite integral. This time the double integral is approximated by boxes which have bases of area \( \Delta x_i \Delta y_j \) and height equal to \( f(x_i, y_j) \). As the number of boxes becomes infinitely large and the area of the base of each box \( A_{ij} \) goes to zero then if the limit exists then we say the function \( f(x, y) \) is integrable and the integral represents the volume under the surface \( f(x, y) \) above the region \( \mathcal{R} \).

A visualization of this concept is depicted in the figure below.

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The Double Integral Can Represent Volume

If the function \( z = f(x, y) \) is always positive over the specific region \( \mathcal{R} \) then the double integral can represent volume.

\[
\int \int_{\mathcal{R}} f(x, y) \, dA = \text{Volume under } f \text{ above the region } \mathcal{R}
\]

The Double Integral Can Represent Area

If the function \( f(x, y) = 1 \) then the double integral can represent area.

\[
\int \int_{\mathcal{R}} f(x, y) \, dA = \int \int_{\mathcal{R}} 1 \, dA = \text{Area of Region } \mathcal{R}
\]

Using an idea from single-variable Calculus, the average value \( \bar{f} \) of a function over a particular region \( \mathcal{R} \) can be found:

\[
\bar{f} = \frac{1}{\text{Area of Region } \mathcal{R}} \int \int_{\mathcal{R}} f(x, y) \, dA = \text{Average value of } f \text{ Over Region } \mathcal{R}
\]