Worksheet 15

TITLE Local Extrema of a Multivariable Function \( f(x, y) \)

CURRENT READING McCallum, Section 15.1

HW #7 (DUE THURSDAY 03/26/15 AT 5PM)

McCallum, Section 15.1: 4, 13, 20, 21, 25, 32, 37, 40*.

McCallum, Section 15.2: 8, 9, 10, 11, 12, 17, 19, 20, 27, 31*, 36.

McCallum, Section 15.3: 2, 5, 8, 14, 18, 21, 31, 34, 44*.

McCallum, Chapter 15 Review: 12, 23, 24, 25, 26, 41, 44*.

SUMMARY

This worksheet discusses the concept of local extrema (maxima or minima) of functions of multivariable functions. We present the definition of critical points for \( f(x, y) \) and introduce the concept of a critical point which is neither a local maximum or local minimum: the saddle point.

DEFINITION: local maximum and local minimum of a multivariable function

\( f \) has a \textbf{local maximum} at the point \( P_0 \) if \( f(P_0) \geq f(P) \), for all points \( P \) near \( P_0 \).

Similarly, \( f \) has a \textbf{local minimum} at the point \( P_0 \) if \( f(P_0) \leq f(P) \), for all points \( P \) near \( P_0 \).

Local Extrema For a Surface \( z = f(x, y) \)

The surface \( z = f(x, y) \) has a \textbf{local maximum} at the point \( (x_0, y_0) \) if \( f(x_0, y_0) \geq f(x, y) \), for all \( (x, y) \) in some neighborhood of \( (x_0, y_0) \).

The surface \( z = f(x, y) \) has a \textbf{local minimum} at \( (x_0, y_0) \) if \( f(x_0, y_0) \leq f(x, y) \), for all \( (x, y) \) in some neighborhood of \( (x_0, y_0) \).

RECALL: Critical Points of \( y = f(x) \)

To find the location of the local maximum or local minimum of a single variable function \( y = f(x) \) you found candidate points called \textbf{critical points} by determining where \( f'(c) = 0 \) or \( f'(c) \) failed to exist. We called the point \( (c, f(c)) \) a critical point of \( f(x) \).

DEFINITION: critical points of a multivariable function

The critical points of a multivariable function \( f(\vec{x}) \) are the points \( \vec{c} \) where the gradient function \( \vec{\nabla} f \) is either \( \vec{0} \) or undefined.

EXAMPLE

\textbf{McCallum, page 832, Example 2.}

Find and analyze any critical points of \( f(x, y) = -\sqrt{x^2 + y^2} \)

QUESTION: What is the difference between a local extrema and a global extrema?
Exercise
McCallum, page 832, Example 3.
Find and analyze any critical points of \( g(x, y) = x^2 - y^2 \). Does this function have a local maximum or local minimum?

**QUESTION:** Does every critical point have to be a local maximum or local minimum?

**Second Derivative Test For Functions Of Two Variables**
Given \((a, b)\) where \(f_x(a, b) = 0\) and \(f_y(a, b) = 0\) Let

\[
D = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b) = \det \begin{bmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{bmatrix}
\]

One can classify the point \((a, b)\) according to the value of \(D\):

- If \(D > 0\) and \(f_{xx}(a, b) > 0\) then \(f(a, b)\) is a local minimum of \(f(x, y)\)
- If \(D > 0\) and \(f_{xx}(a, b) < 0\) then \(f(a, b)\) is a local maximum of \(f(x, y)\)
- If \(D < 0\) then \(f(a, b)\) is a saddle point of \(f(x, y)\)
- \(D = 0\) the test is inconclusive so that \(f(a, b)\) could be a local maximum, local minimum, a saddle point or none of the above!

**EXAMPLE**
Use the Second Derivative Test to classify the critical points of \( f(x, y) = Ax^2 + Bxy + Cy^2 \) based on the values of \(A\), \(B\) and \(C\).
GROUP WORK

McCallum, page 835, Example 6. Find the local maxima, local minima and saddle points of 
\[ f(x, y) = \frac{1}{2}x^2 + 3y^3 + 9y^2 - 3xy + 9y - 9x. \]

McCallum, page 836, Example 7. Classify the critical points of 
\[ f(x, y) = x^4 + y^4, \]
\[ g(x, y) = -x^4 - y^4 \]
and 
\[ h(x, y) = x^4 - y^4. \]