Math 212 Spring 2015 **SHOW ALL YOUR WORK** Quiz **TEN**

1. Consider the vector field $\vec{F}(x,y) = (1 - ye^{-x})\hat{i} + e^{-x}\hat{j}$.

(a) *(5 points.)* Show by direct computation that the line integral of $\vec{F}$ along the straight line path from $(0,1)$ to $(1,2)$ is $2e^{-1}$. **[HINT:** $\int ue^u\,du = ue^u - e^u$]

STEP 1
parametrize path

$x = t$
$y = 1+t$
$t: 0 \to 1$

STEP 2
compute integrand

$\dot{x} = 1$
$\dot{y} = 1$

$\vec{F}(x,y) = \begin{pmatrix} 1-(1+t)e^{-t} \\ e^{-t} \end{pmatrix}$

STEP 3
Integrate!

$\int_0^1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1-(1+t)e^{-t} \\ e^{-t} \end{pmatrix} dt$

$I = \int_0^1 1 - e^{-t} - te^{-t} + e^{-t}\,dt = \int_0^1 1 - te^{-t}\,dt$

$u = -t$
$du = -dt$
$t=0, u=0$
$t=1, u=-1$

$I = \int_0^1 1\,dt + \int_0^{-1} (-t)e^{-t}\,dt$

$I = t\Big|_0^1 + \int_0^{-1} ue^u\,du$

$= 1 + \int_{-1}^{0} ue^u\,du = 1 + ue^u - e^u\Big|_{-1}^{0}$

$= 1 + 0 - e^0 - (-1e^{-1} - e^{-1})$

$I = 1 - 1 - (-2e^{-1}) = \boxed{2e^{-1}}$

(b) *(3 points.)* Show that this vector field is a **gradient field**, in other words that there exists a function $f(x,y)$ such that $\vec{\nabla} f = \vec{F}(x,y) = (1 - ye^{-x})\hat{i} + e^{-x}\hat{j}$. **[HINT: find $f(x,y)$ so that $\vec{\nabla} f = \vec{F}$.]**

If $\vec{F}$ is a ~~vector~~ gradient field $\Longleftrightarrow$ $\text{curl}\,\vec{F} = \vec{0}$

$\text{curl}\,\vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}(e^{-x}) - \frac{\partial}{\partial y}(1 - ye^{-x}) = -e^{-x} - (-e^{-x}) = 0$

$f_x = 1 - ye^{-x}$, $f_y = e^{-x} \Rightarrow f = ye^{-x} + A(x) \Rightarrow f_x = -ye^{-x} + A'(x) = -ye^{-x} + 1$

$A'(x) = 1$
$A(x) = x + C$

$\boxed{f = ye^{-x} + x + C}$

$\vec{\nabla} f = (-ye^{-x} + 1)\hat{i} + e^{-x}\hat{j} = \vec{F}$

(c) *(2 points.)* Use the Fundamental Theorem for Line Integrals (and your answer in **(b)**) to show that the value of the line integral of $\vec{F}$ from $(0,1)$ to $(1,2)$ is the same value regardless of the path taken to travel between these two points. **What is this value and how is it related to f you found in (b)?**

By FTLI where $\vec{F} = \vec{\nabla} f$

$\int_{(0,1)\to(1,2)} \vec{F} \cdot d\vec{x} = f(1,2) - f(0,1) = f(\vec{x}_B) - f(\vec{x}_A)$

$= 2e^{-1} + 1 - (1e^0 + 0)$

$= 2e^{-1}$ ✓

Since $\vec{F}$ is gradient field $\Leftrightarrow$ $\vec{F}$ is path-independent so $\int_{\vec{x}_A}^{\vec{x}_B} \vec{F} \cdot d\vec{x}$ is same