1. Our goal is to find the maximum and minimum values of the surface $z = f(x, y) = x^2+y^2-x-y+1$ constrained to the interior of the unit disk $D : x^2 + y^2 \leq 1$.

(a) (*3 points*) Show that the only critical point of $f(x, y)$ occurs at $(1/2, 1/2, 1/2)$.

$f_x = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

$f_y = 2y - 1 = 0 \quad y = 1/2$

$f(1/2, 1/2) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - \frac{1}{2} - \frac{1}{2} + 1$

$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is a critical point

(b) (*2 points*) Show that since the boundary can be parametrized by the curve $\vec{x}(t) = (\cos(t), \sin(t))$ with $0 \leq t \leq 2\pi$ the surface intersected with the boundary becomes a curve $g(t) = f(x(t), y(t)) = 2 - \sin t - \cos t$.

$g(t) = \cos^2 t + \sin^2 t - \cos t - \sin t + 1$

$= 1 + 1 - \sin t - \cos t$

$= 2 - \cos t - \sin t$

$0 \leq t \leq 2\pi$

(c) (*3 points*) Explain why $g(t)$ must attain its extreme values at either $t = 0$, $t = \pi/4$, $t = 5\pi/4$ or $t = 2\pi$.

$g'(t) = +\sin t - \cos t = 0 \Rightarrow \tan t = 1$

$\Rightarrow t = \pi/4$ or $5\pi/4$

Since $g(t)$ is continuous on a closed bounded set it must have a global max & global min on $0 \leq t \leq 2\pi$. They must occur at critical points $t = 0$, $t = \frac{\pi}{4}$, $t = \frac{5\pi}{4}$ or $t = 2\pi$

(d) (*2 points*) Use information from (a), (b) and (c) to write down the coordinates of the maximum and minimum values of the surface $z = f(x, y)$ where the input values must lie on $D : x^2 + y^2 \leq 1$.

$g(0) = 1$

$g(2\pi) = 1$

$g(\pi/4) = 2 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 2 - \sqrt{2}$

$g(5\pi/4) = 2 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 2 + \sqrt{2}$

$f(1/2, 1/2) = 1/2 \leftarrow$ GLOBAL MIN

GLOBAL MAX

$\rightarrow \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 2 + \sqrt{2}\right)$