(Adapted from Math 212 Spring 2006 Midterm #2.)

1. Consider the function \( u(x, y, z) = f(x - y, y - z, z - x) \). Our goal is to show that a function \( u \) with this form satisfies the following famous partial differential equation

\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.
\]

(a) (3 points.) Consider a function \( u = f(r, s, t) \) where \( r = r(x, y, z) \), \( s = s(x, y, z) \) and \( t = t(x, y, z) \) are given. In other words, although \( u \) is a function of \( r, s \) and \( t \), since each of these functions is a function of \( x, y \) and \( z \) one can consider \( u \) as a function of \( x, y \) and \( z \). Draw a “tree diagram” reflecting the relationships between the variables.

(b) (3 points) Use the Multivariable Chain Rule to write down expressions for \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \) and \( \frac{\partial u}{\partial z} \) where \( u \) is assumed to be related to \( x, y \) and \( z \) as described in part (a).

\[
\begin{align*}
\frac{\partial u}{\partial x} &= u_r \frac{\partial r}{\partial x} + u_s \frac{\partial s}{\partial x} + u_t \frac{\partial t}{\partial x}, \\
\frac{\partial u}{\partial y} &= u_r \frac{\partial r}{\partial y} + u_s \frac{\partial s}{\partial y} + u_t \frac{\partial t}{\partial y}, \\
\frac{\partial u}{\partial z} &= u_r \frac{\partial r}{\partial z} + u_s \frac{\partial s}{\partial z} + u_t \frac{\partial t}{\partial z}.
\end{align*}
\]

(c) (4 points) Let \( r = x - y, s = y - z \) and \( t = z - x \). Use this information and your answer to (b) to show that \( u(x, y, z) = f(x - y, y - z, z - x) \) satisfies the equation \( u_x + u_y + u_z = 0 \).

\[
\begin{align*}
r_x &= 1 & s_x &= 0 & t_x &= -1, \\
r_y &= -1 & s_y &= 1 & t_y &= 0, \\
r_z &= 0 & s_z &= -1 & t_z &= 1.
\end{align*}
\]

\[
\begin{align*}
u_x &= u_r - u_t &= u_r \cdot 1 + u_s \cdot 0 + u_t \cdot -1, \\
u_y &= -u_r - u_s &= u_r \cdot -1 + u_s \cdot 1 + u_t \cdot 0, \\
u_z &= -u_s + u_t &= u_r \cdot 0 + u_s \cdot -1 + u_t \cdot 1, \\
u_x + u_y + u_z &= u_r - u_t - u_r + u_s - u_s + u_t \\
&= 0.
\end{align*}
\]