1. The goal of this problem is to show that the maximum volume of a rectangular box without a lid which can be made from 12 m$^2$ of cardboard is 4 m$^3$.

Let $x$, $y$, and $z$ be the length, width and height of the box, respectively.

a. (1 point) Write down the objective function $f(x, y, z)$ to be optimized.

$$f(x, y, z) = xyz$$

Maximizing Volume: Objective

b. (1 point) Write down the constraint function $g(x, y, z)$.

$$g(x, y, z) = xy + 2yz + 2xz = 12$$
The surface area of box is fixed (constraint)

c. (2 points) Using the method of Lagrange Multipliers, write down the set of equations that need to be solved in order to maximize $f$ subject to the constraint $g$. How many equations in how many unknowns will you have?

$$\nabla f = \lambda \nabla g$$
$$g = 12$$

$$x + 2z = \lambda \cdot z$$
$$y + 2z = \lambda \cdot z$$
$$2y + 2x = \lambda \cdot x$$
$$xy + 2yz + 2xz = 12$$

4 equations in 4 unknowns

d. (1 point) Find the dimensions of the rectangular box without lid with maximum volume with surface area 12 m$^2$.

$$xy + 2xz = \lambda \cdot xy \cdot 2z = xy + 2xy = 2yz + 2xz$$

$$\Rightarrow 2xz = 2yz \Rightarrow x = 2y$$

$$2xz + 2yz + 2z^2 = 12$$

$$6z^2 = 12 \Rightarrow z = 2$$

$$x = \frac{4}{3}, y = \frac{2}{3}, z = 1$$

Volume is

$$xyz = 2 \cdot 2 \cdot 1 = 4$$