Exam 1: Multivariable Calculus

Math 212 Fall 2014
Prof. Ron Buckmire

Friday October 3
11:45am-12:40pm

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Directions:

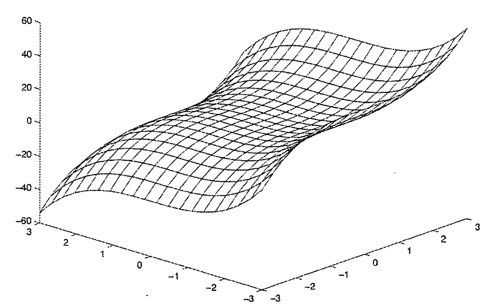
Read *all* problems first before answering any of them. This tests consists of three (3) problems (and a BONUS problem) on six (6) pages.

The topic of the problem is **in bold**, the number of points each problem is worth is in *italics* and the kind of skills required to solve each problem are in ALL CAPS.

This is a 55-minute, open notes, open book, test. **No calculators or electronic devices may be used.**

You must show all relevant work to support your answers. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your "scratch work."

Questions Policy: FEEL FREE TO ASK CLARIFICATION QUESTIONS AT ANY TIME!



No.	Score	Maximum
1		30
2		30
3		40
BONUS		10
Total		**100**

1. (*30 points.*) ANALYTIC, COMPUTATIONAL, VISUAL. **Equation of Planes, Vector Operations.**

1(a) (*15 points.*) Find the general equation of the plane $\mathcal{P}$ in $\mathbb{R}^3$ that contains all three points $A(1,0,0)$, $B(1,2,-2)$, and $C(0,-3,4)$. ALSO EXPLAIN IN WORDS YOUR TECHNIQUE FOR DETERMINING THE EQUATION OF THE PLANE $\mathcal{P}$.

Find the displacement vectors $\vec{AB}$ and $\vec{AC}$ that lie in the plane. Take their cross-product to find normal vector. Use this to compute general equation

$$\vec{AB} = \begin{pmatrix}1\\2\\-2\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\2\\-2\end{pmatrix} \qquad \vec{AC} = \begin{pmatrix}0\\-3\\4\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\-3\\4\end{pmatrix}$$

$$\vec{n} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 2 & -2\\ -1 & -3 & 4\end{vmatrix} = \hat{i}(8-6) - \hat{j}(0-2) + \hat{k}(0+2) = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\vec{n}\cdot\vec{x} = \vec{n}\cdot\vec{p} \Rightarrow 2x+2y+2z = \begin{pmatrix}2\\2\\2\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix} = 2 \Rightarrow \boxed{x+y+z=1}$$

1(b) (*15 points.*) What is the minimum distance between the plane $\mathcal{P}$ found in part **1(a)** and the plane $\mathcal{Q}$ given by the equation $x+y+z=0$? ALSO EXPLAIN IN WORDS YOUR TECHNIQUE FOR DETERMINING THE MINIMUM DISTANCE BETWEEN THE PLANES.

Since the normal of P is the same as the normal of Q these planes are parallel, so there is a non-zero distance between them (since they are not the same plane.)

The distance is found by finding the magnitude of the displacement vector between any point on plane P and any point on plane Q, projected in the direction of the normal.

$$\vec{n} = \hat{i} + \hat{j} + \hat{k}$$

$$\vec{d} = \begin{pmatrix}1\\2\\-2\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix} = \hat{i} + 2\hat{j} - 2\hat{k}$$

$$\vec{p} = \mathrm{proj}_{\vec{n}}(\vec{d}) = \left(\frac{\vec{d}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\right)\vec{n} = \frac{\begin{pmatrix}1\\2\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}}{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Distance between planes is

$$\|\vec{p}\| = \frac{1}{3}\sqrt{1+1+1} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

2. (*30 points.*) VERBAL, ANALYTIC, COMPUTATIONAL. **Gradient, Directional Derivative, Tangent Planes** Consider the following information about a mystery function $M(x, y, z)$:

(1) $M(x, y, z)$ increases most rapidly in the direction $\hat{v} = \frac{1}{7}(3\hat{\imath}+6\hat{\jmath}+2\hat{k})$ at the point P$(-4, 2, 4)$.

(2) At the point P$(-4, 2, 4)$ the maximum rate of change of $M(x, y, z)$ is exactly 5.

For each of the following questions, try to do the requested calculation or explain what missing information is necessary in order to do the calculation and how you would do the requested calculation if you had that information.

2(a) (*10 points.*) What is $M_{\vec{v}}(-4, 2, 4)$, the rate of change of $M(x, y, z)$ at the point P$(-4, 2, 4)$ in the direction $\vec{v} = 4\hat{\imath} - 3\hat{k}$?

At $(-4,2,4)$, $\|\vec{\nabla} M\| = 5$ and $\vec{\nabla} M$ is in direction of $\vec{v}$.

$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{5}(4\hat{\imath} - 3\hat{k})$

$\vec{\nabla} M = \frac{1}{7}(3\hat{\imath} + 6\hat{\jmath} + 2\hat{k}) \cdot 5 = \frac{15}{7}\hat{\imath} + \frac{30}{7}\hat{\jmath} + \frac{10}{7}\hat{k}$

$$M_{\vec{v}}(-4,2,4) = \vec{\nabla} M(-4,2,4) \cdot \frac{1}{5}(4\hat{\imath} - 3\hat{k}) = \left(\frac{15}{7}\hat{\imath} + \frac{30}{7}\hat{\jmath} + \frac{10}{7}\hat{k}\right) \cdot \left(\frac{4}{5}\hat{\imath} - \frac{3}{5}\hat{k}\right)$$

$$= \frac{15}{7} \cdot \frac{4}{5} + \frac{10}{7} \cdot \frac{(-3)}{5} = \frac{30}{35} = \frac{6}{7}$$

2(b) (*10 points.*) In what direction $\vec{u}$ does the mystery function $M(x, y, z)$ change the least in magnitude at the point $(-4, 2, 4)$, i.e. in what direction parallel to $\vec{u}$ is $M_{\vec{u}}(-4, 2, 4)$ exactly zero? IF THERE IS MORE THAN ONE SUCH DIRECTION, EXPLAIN WHY AND STATE HOW MANY DIRECTIONS $M_{\vec{u}}(-4, 2, 4) = 0$.

$M(x,y,z)$ will be constant, i.e. $M_{\vec{u}}(-4,2,4) = 0$ in all directions perpendicular to $\frac{1}{7}(3\hat{\imath} + 6\hat{\jmath} + 2\hat{k})$. That is an infinite number of vectors that ~~lay~~ lie in a plane normal to $\frac{1}{7}(3\hat{\imath} + 6\hat{\jmath} + 2\hat{k})$.

The vectors will lie in the plane $3x + 6y + 2z = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} = 8$

2(c) (*10 points.*) What is the general equation of the tangent plane to the mystery function $M(x, y, z)$ at the point P$(-4, 2, 4)$?

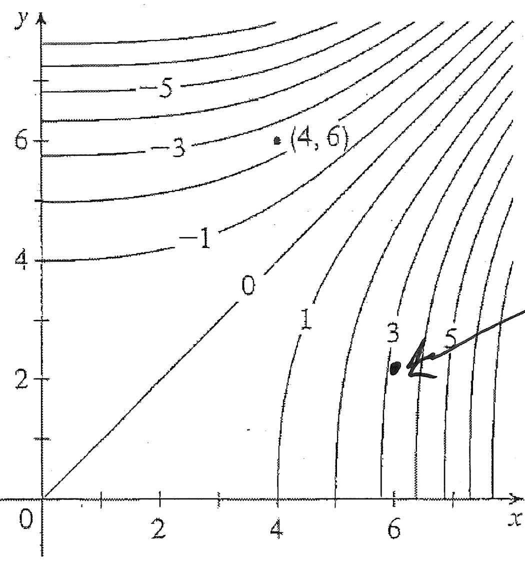
You can't find a tangent plane to $M(x,y,z)$ since this is a four-dimensional object. You CAN find the tangent plane to the LEVEL SETS of $M(x,y,z) = K$ at $P(-4,2,4)$

It would look like

$$\vec{\nabla} M(-4,2,4) \cdot \vec{x} = \vec{\nabla} M(-4,2,4) \cdot -4\hat{\imath} + 2\hat{\jmath} + 4\hat{k}$$

$$3x + 6y + 2z = 0$$

3. (*40 points.*) ANALYTIC, VISUAL, VERBAL. **Partial Derivatives, Multivariable Functions, Cross-sections, Level Sets.** Consider the following figure which depicts the contour diagram of an unknown function $f(x, y)$. **NOTE:** $f(x, y) = 0$ **when** $y = x$.



3(a) (*10 points.*) What are the signs of $f_x(4, 6)$ and $f_y(4, 6)$? EXPLAIN YOUR ANSWER.

$f_x(4,6) > 0$ because f increases as x increases near 4.

$f_y(4,6) < 0$ because f decreases as y increases near 6

3(b) (*10 points.*) Indicate on the figure above a location (a, b) in the xy-plane where $f_x(a, b)$ **and** $f_y(a, b)$ will both be greater than they are at $(4, 6)$. Give the approximate coordinates of this point (if it exists). EXPLAIN YOUR ANSWER.

Near $(6,2)$ f_x will be larger (and positive) than $f_x(4,6)$ because contours are closer together.

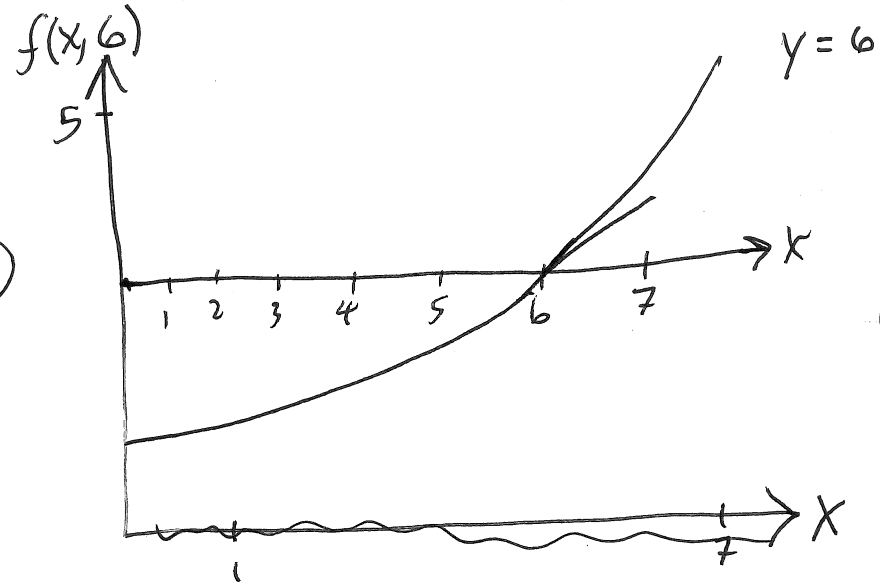
Near $(6,2)$ f_y will be nearly zero (less negative than it was at $(4,6)$.

3(c) (*20 points.*) Again consider the contour diagram of the unknown function $f(x, y)$ given on the previous page.

In the space below give an accurate sketch of the graphs of the cross-sections of the unknown function $f(x, y)$. Sketch the cross-section of $f(x, y)$ when $x = 4$ from $0 \leq y \leq 7$. Then on a different set of labeled axes, give the cross-section $f(x, y)$ when $y = 6$ from $0 \leq x \leq 7$.

In the rest of the space, give written explanations for important features of your graphs (i.e. any changes in sign, slope or concavity, location of any zeroes, maximum and minimum values).

(6)



f(x, 6) is always concave-up

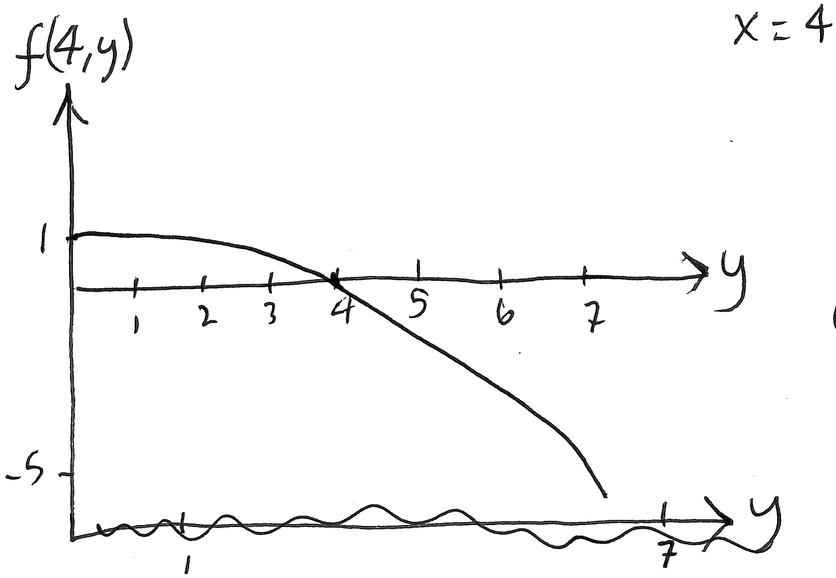
Has zero slope near x=0

(4) Has max value of 5+ near x = 7.

Has min value of -3 near x=0

Root is at x=6

(6)



f(4, y) is concave down on 0<y<7

(4) Has root at y=4

Has max near y=0 of ~1.

Has min value of ~-5 near y=7

Slope of zero near y=0.

BONUS QUESTION. Limits, Continuity. *(10 points.)*
Consider $g : \mathbb{R}^2 \to \mathbb{R}$ where

$$g(x, y) = \begin{cases} \dfrac{x^2y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ A, & (x, y) = (0, 0) \end{cases}$$

If it is possible, choose a value for the unknown constant A which will make the function $g(x, y)$ continuous for every $(x, y) \in \mathbb{R}^2$.

EXPLAIN YOUR ANSWER THOROUGHLY AND SHOW ALL YOUR WORK, EXTRA CREDIT POINTS ARE HARD TO EARN!

$g(x,y)$ is continuous for all $(x,y) \neq (0,0)$ since it is the ratio of two polynomials.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0$$

~~Let x~~ when $x^2+y^2 < \delta$ we can show $x^2 < x^2+y^2 < \delta$ and $y^2 < x^2+y^2 < \delta$

$$\frac{x^2y^2}{x^2+y^2} < \frac{(x^2+y^2)(x^2+y^2)}{(x^2+y^2)} = x^2+y^2 < \delta$$

so as $\delta \to 0$ (i.e. (x,y) gets closer to $(0,0)$) the value of $\frac{x^2y^2}{x^2+y^2}$ gets closer to zero.

Let $A = 0$.

Then $\lim_{(x,y)\to(0,0)} g(x,y) = g(0,0)$ so $g(x,y)$ will be continuous at $(x,y) = (0,0)$.