Multivariable Calculus

Math 212 §2 Fall 2014 | Fowler 309 MWF 11:45am - 12:40pm

 | http://faculty.oxy.edu/ron/math/212/14/

Worksheet 24

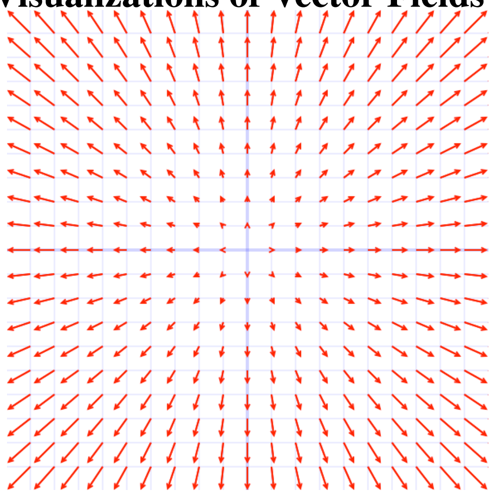
TITLE Vector Fields and the Line Integral
CURRENT READING McCallum, Section 17.3, 18.1-18.2)
HW #11 (DUE Wednesday 11/12/14 5PM)
McCallum, *Section 17.2*: 1, 4, 10, 14, 27, 28, 36.
McCallum, *Chapter 17.3*: 1, 12,18, 21, 28, 29.
McCallum, *Chapter 17 Review*: 2, 4, 5, 7, 10, 12, 18, 20, 32, 33, 34.

SUMMARY
This worksheet introduces the idea of vector fields, the concept of a line or path integral and the algorithm for how to evaluate line integrals given a path.

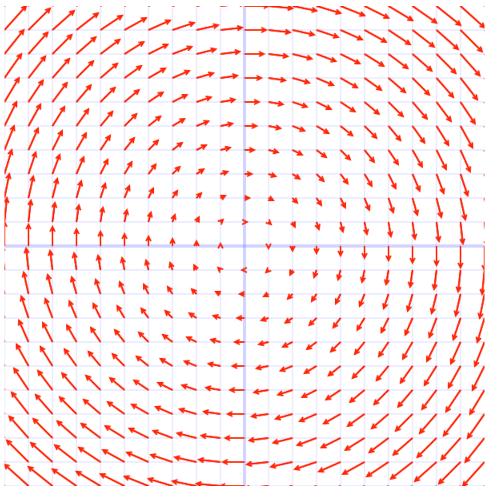
DEFINITION: vector field in $\mathbb{R}^2$

A **vector field** in $\mathbb{R}^2$ is a function $\vec{F}(x, y)$ which produces as its output a 2-dimensional vector at every point in the plane. For example, the gradient of a function grad $f(x, y)$ is a vector which points in the direction of greatest increase of f at every position in the xy-plane.

Visualizations of Vector Fields



$\vec{F}(x, y) = x\hat{i} + y\hat{j}$



$\vec{G}(x, y) = y\hat{i} - x\hat{j}$

Exercise

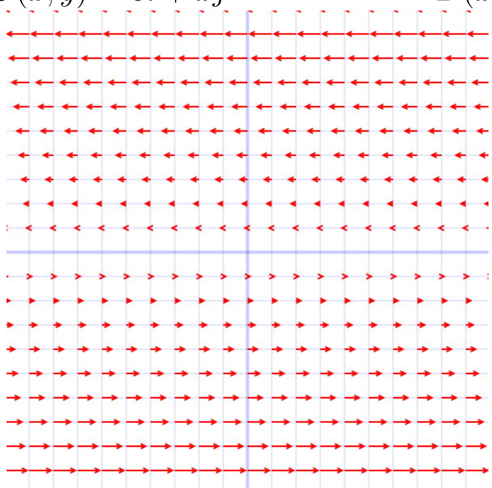
Give a sketch of the vector field $\vec{H}(x, y) = -x\hat{i}$

Exercise

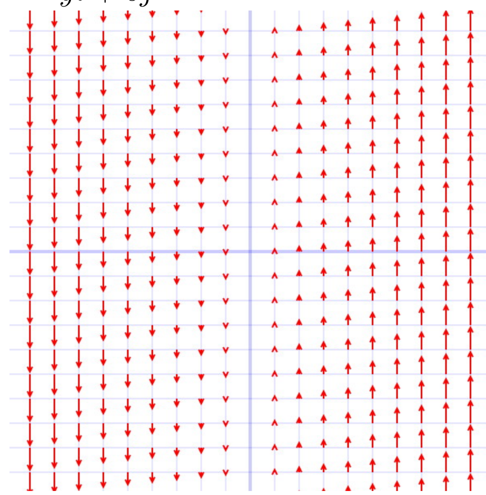
Match the vector field with the corresponding figure.

$\vec{A}(x,y) = (x+y)\hat{\imath} + (x-y)\hat{\jmath}$ $\quad$ $\vec{B}(x,y) = \dfrac{-x}{\sqrt{x^2+y^2}}\hat{\imath} + \dfrac{-y}{\sqrt{x^2+y^2}}\hat{\jmath}$

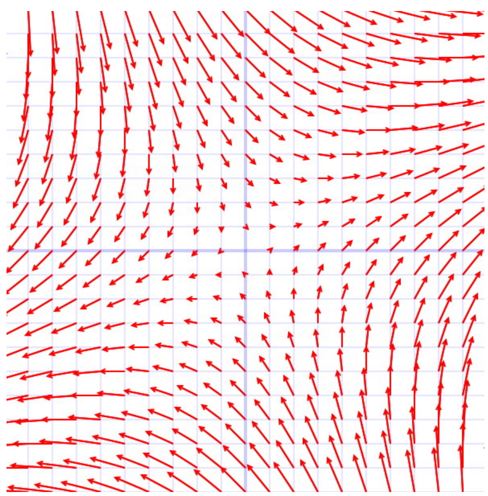
$\vec{C}(x,y) = 0\hat{\imath} + x\hat{\jmath}$ $\quad$ $\vec{D}(x,y) = -y\hat{\imath} + 0\hat{\jmath}$



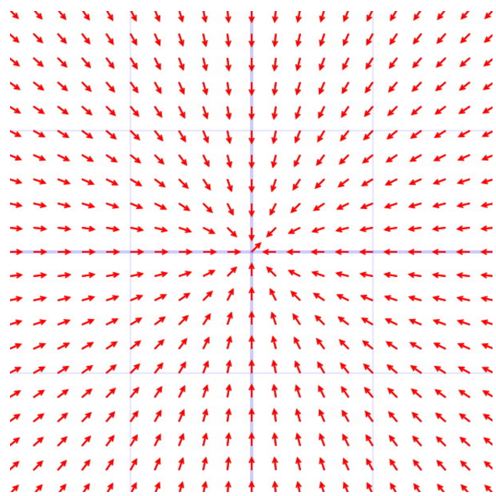
FIELD 1



FIELD 2



FIELD 3



FIELD 4

Gradient Field

If a vector field is formed from taking the gradient of some scalar function f, then such a field is known as a **gradient field** and is given by $\vec{F} = \vec{\nabla} f$.

GROUPWORK

Determine whether any of the given fields $\vec{F} = x\hat{\imath} + y\hat{\jmath}$, $\vec{G} = -y\hat{\imath} + x\hat{\jmath}$ or $\vec{H} = x\hat{\imath}$ are gradient fields.

Path Integrals or Line Integrals

DEFINITION: path integral or line integral

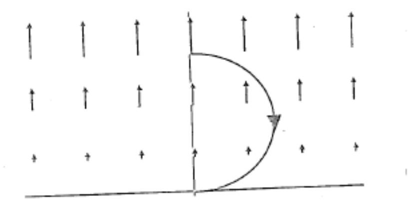
Given a vector field $\vec{F}:\mathbb{R}^2 \to \mathbb{R}^2$ and a path C in $\mathbb{R}^2$ given by the parametrization $\vec{g}(t):\mathbb{R}\to\mathbb{R}^2$ for $a \le t \le b$, the **path integral** of $\vec{F}$ over C is written as $\int_C \vec{F}\cdot d\vec{x}$ and evaluated by computing $\int_a^b \vec{F}(\vec{g}(t))\cdot \frac{d\vec{g}}{dt}\, dt$.

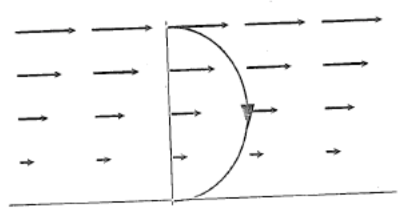
CONCEPTUAL UNDERSTANDING

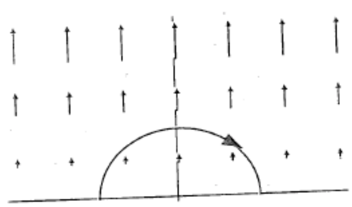
The meaning of the expression $\int_C \vec{F}\cdot d\vec{x}$ is the amount of work done by a Force $\vec{F}(\vec{x})$ acting along the path C.

EXAMPLE

Consider the following paths and vector fields. Let's use our understanding of path integrals to at least estimate the sign of the line integral $\int_C \vec{F}\cdot d\vec{x}$.







Properties of Path Integrals or Line Integrals

Given a scalar constant λ, vector fields $\vec{F}$ and $\vec{G}$ and oriented curves (i.e. paths) C_1, C_2 and C the following properties apply to line integrals.

Linearity

$$\int_C (\vec{F}+\vec{G})\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} + \int_C \vec{G}\cdot d\vec{r}$$

$$\int_C \lambda\vec{F}\cdot d\vec{r} = \lambda\int_C \vec{F}\cdot d\vec{r}$$

Reversing Orientation

$$\int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}$$

Additivity

$$\int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_1+C_2} \vec{F}\cdot d\vec{r}$$

Evaluation of Path Integrals or Line Integrals

ALGORITHM

This is an algorithm (step-by-step method) for evaluating the line integral $\int_C \vec{F} \cdot d\vec{r}$

STEP 1 Parametrize the path C with a function of your choice $\vec{r}(t)$ for $a \le t \le b$

STEP 2 Compute $\dfrac{d\vec{r}}{dt}$ and $\vec{F}(\vec{r}(t))$ and take their dot product. i.e. $\vec{F}(\vec{r}(t)) \cdot \dfrac{d\vec{r}}{dt}$

STEP 3 Integrate the function of t given in Step 2 with respect to t from $t = a$ to $t = b$, i.e. the number $\displaystyle\int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$ is the value of the given line integral

EXAMPLE

McCallum, Example 2, page 959. Given the vector field $\vec{F} = x\hat{i} + y\hat{j}$ evaluate $\int_C \vec{F} \cdot d\vec{r}$ when
(a) C_1 is the line segment joining $(1, 0)$ to $(0, 2)$
(b) C_2 is a path formed by part of a parabola with its vertex at $(0, 2)$ joining $(1, 0)$ to $(0, 2)$
(c) C_3 is the closed path formed by following the triangle with vertices at $(0, 0)$, $(1, 0)$ and $(0, 2)$ traversed in a counter-clockwise fashion.