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# Multivariable Calculus

Math 212 §2 Fall 2014  
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Fowler 309 MWF 11:45am - 12:40pm  
<http://faculty.oxy.edu/ron/math/212/14/>

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## Worksheet 21

**TITLE** Evaluating Multiple Integrals Using Other Coordinate Systems

**CURRENT READING** McCallum, Section 16.4-16.5, 21.2

**HW #8 (DUE Wednesday 11/4/14 5PM)**

McCallum, *Section 16.3*: 2, 5, 6, 28, 39, 40, 41, 42, 54\*, 55\*.

McCallum, *Chapter 16.4*: 3, 7, 8, 17, 20, 22.

McCallum, *Chapter 16.5*: 12, 13, 14, 15, 21, 22, 23, 63\*, 73.

McCallum, *Chapter 16 Review*: 1, 4, 10, 11, 12, 14, 20, 23, 55\*, 56\*.

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### SUMMARY

This worksheet discusses how to compute iterated integrals in other coordinate systems, namely polar coordinates, spherical coordinates and cylindrical coordinates.

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**RECALL** Points in the  $xy$ -plane can also be represented by a different coordinate system, called **polar coordinates** where  $r = \sqrt{x^2 + y^2}$  and  $\theta = \arctan(y/x)$ . In other words,

$$x = r \cos \theta, \quad y = r \sin \theta$$

### The Double Integral In Polar Coordinates

Consider the following integral

$$\int_R f(x, y) dA = \int_\alpha^\beta \int_a^b f(r, \theta) r dr d\theta \quad (1)$$

### NOTE

Note that the  $dA$  which in regular Cartesian coordinates is either  $dx dy$  or  $dy dx$  becomes  $r dr d\theta$  NOT simply  $dr d\theta$ !

Some problems are easier to do in polar coordinates than cartesian coordinates!

### EXAMPLE

Evaluate  $\int_D \cos(x^2 + y^2) dA$  where  $D$  is the disk (i.e. interior and boundary) of radius  $\sqrt{\pi/2}$  centered at  $(0, 0)$ .

**THEOREM****Jacobi's Theorem for Transforming Integrals Between Coordinate Systems**

The integral of a continuous function  $f(\vec{x})$  over a region  $\mathcal{W}$  in  $\mathbb{R}^n$  can be transformed into an equivalent integral of  $f(\vec{T}(\vec{x}))$  in a region  $\mathcal{W}^*$  where  $\vec{T}$  is a continuously differentiable transformation that maps  $\mathcal{W}$  to  $\mathcal{W}^*$ , i.e.  $\mathcal{W}^* = T(\mathcal{W})$ .

In other words, suppose in  $\mathbb{R}^3$  that  $\vec{T}(\vec{x}) = \begin{bmatrix} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{bmatrix}$  so that

$$\iiint_{\mathcal{W}} f(x, y, z) dx dy dz = \iiint_{\mathcal{W}^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw \text{ in } \mathbb{R}^3$$

and in  $\mathbb{R}^2$   $\vec{T}(\vec{x}) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$  so that

$$\iint_{\mathcal{W}} f(x, y) dx dy = \iint_{\mathcal{W}^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \text{ in } \mathbb{R}^2$$

The expressions  $\left| \frac{\partial(x, y)}{\partial(u, v)} \right|$  and  $\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right|$  are called the **Jacobian** of the transformation. In actuality they are the determinant of the Jacobian matrix associated with the transformation.

**DEFINITION: The Jacobian Matrix**

The Jacobian matrix of a function  $\vec{f}(\vec{x})$  is a matrix where the term in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column is the expression  $\frac{\partial f_i}{\partial x_j}$  where  $f_i$  is the  $i^{\text{th}}$  component of the vector function  $\vec{f}(\vec{x})$  and  $x_j$  is the  $j^{\text{th}}$  component of the vector variable  $\vec{x}$ .

The Jacobian matrix for  $\vec{T}$  in  $\mathbb{R}^3$  and  $\mathbb{R}^2$  respectively is given below

$$\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \quad \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

**CONCEPTUAL UNDERSTANDING**

Generally, we use Jacobi's theorem to convert from Cartesian coordinates to polar, spherical, and cylindrical co-ordinates.

**Change of Variables: Polar Coordinates**

$$\iint_D f(x, y) dx dy = \iint_{D^*} f(r \cos \theta, r \sin \theta) r dr d\theta$$

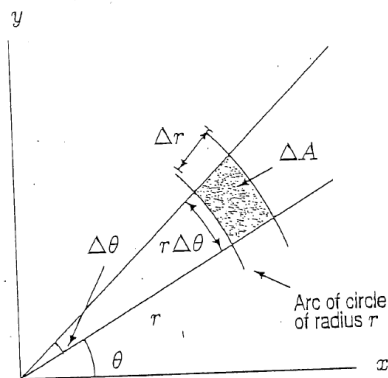
**Change of Variables: Cylindrical Coordinates**

$$\iiint_{\mathcal{W}} f(x, y, z) dx dy dz = \iiint_{\mathcal{W}^*} f(r \cos \theta, r \sin \theta, z) r dr d\theta dz$$

**Change of Variables: Spherical Coordinates**

$$\iiint_{\mathcal{W}} f(x, y, z) dx dy dz = \iiint_{\mathcal{W}^*} f(r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi) r^2 \sin \phi dr d\theta d\phi$$

### Visualizing The Area Differential In Polar Coordinates



The area of the segment of the circular arc of radius  $r$  of angular width  $\Delta\theta$  and length  $\Delta r$  is  $\Delta A \approx (r\Delta\theta)(\Delta r)$

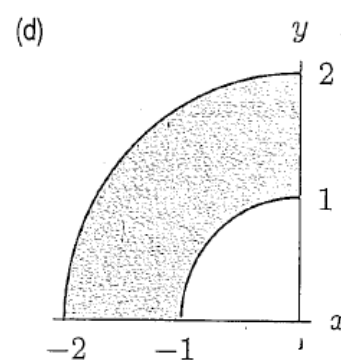
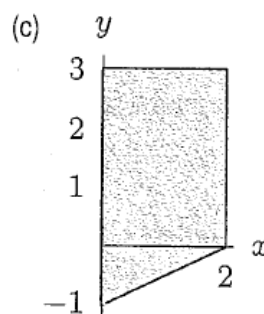
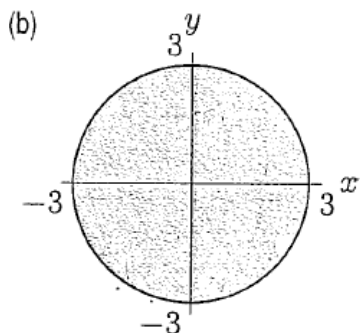
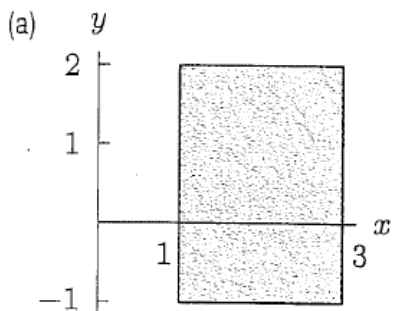
**EXAMPLE**

We can use the Jacobian of the transformation from Cartesian coordinates  $(x, y)$  to polar coordinates  $(r, \theta)$  to explain why  $dx dy = r dr d\theta$ .

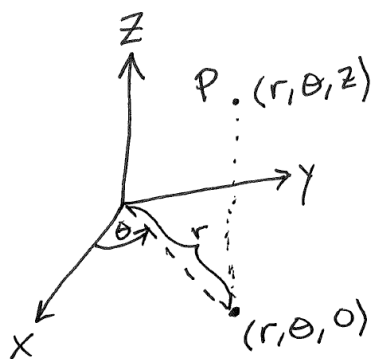
Let  $\vec{T}(\vec{x}) = \begin{bmatrix} x(r, \theta) \\ y(r, \theta) \end{bmatrix}$  where  $x(r, \theta) = r \cos \theta$  and  $y(r, \theta) = r \sin \theta$  and compute  $\left| \frac{\partial(x, y)}{\partial(r, \theta)} \right|$

**Exercise**

**McCallum, page 893, Example 3.** For each of the regions below decide whether to integrate using polar or Cartesian coordinates. Write down an iterated integral of an arbitrary function  $f(x, y)$  over the given region.

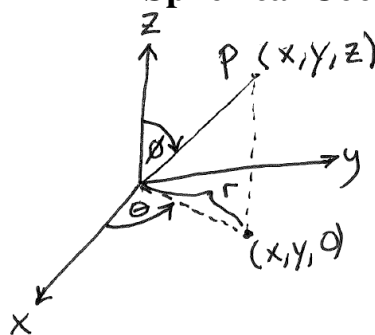


### Cylindrical Coordinates



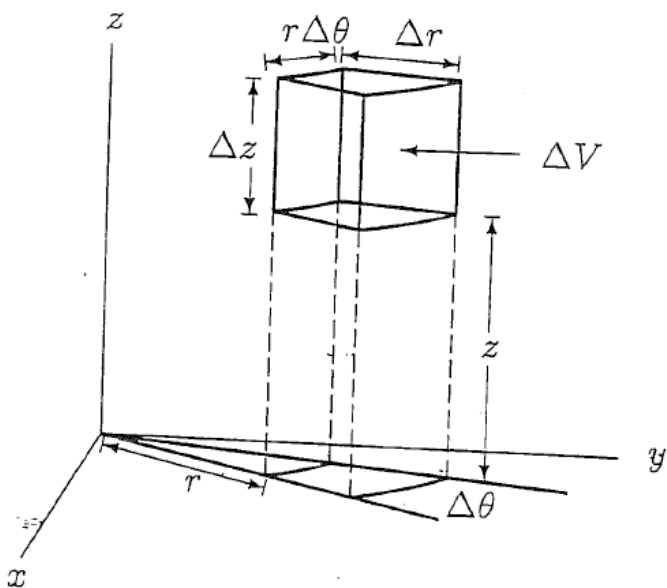
$$\begin{aligned}
 x &= r \cos \theta \\
 y &= r \sin \theta \\
 z &= z \\
 0 &\leq r < \infty, -2\pi \leq \theta \leq 2\pi, -\infty < z < \infty \\
 r^2 &= x^2 + y^2
 \end{aligned}$$

### Spherical Coordinates

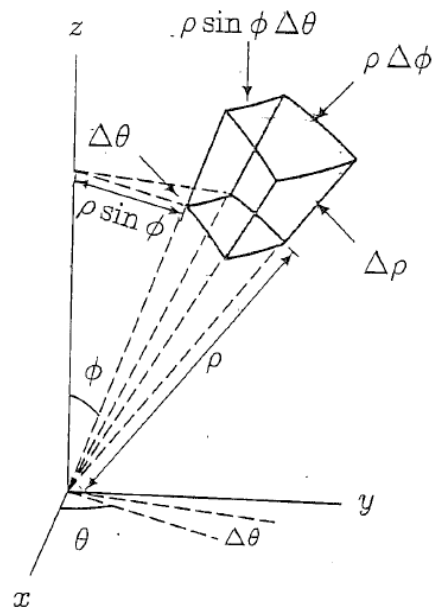


$$\begin{aligned}
 x &= \rho \sin \phi \cos \theta \\
 y &= \rho \sin \phi \sin \theta \\
 z &= \rho \cos \phi \\
 0 &\leq \rho < \infty, -2\pi \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi \\
 \rho^2 &= x^2 + y^2 + z^2
 \end{aligned}$$

### Visualizing The Volume Differential In Cylindrical Coordinates and Spherical Coordinates



$$\Delta v \approx r \Delta r \Delta z \Delta \theta$$



$$\Delta V \approx \rho^2 \sin \phi \Delta \rho \Delta \theta \Delta \phi$$

**EXAMPLE**

**McCallum, page 903, Exercise 19.** Write a triple integral in cylindrical coordinates giving the volume of a sphere of radius  $K$  centered at the origin. Use the order  $dz dr d\theta$ .

Evaluate the triple integral to show the volume of the sphere of radius  $K$  is  $\frac{4}{3}\pi K^3$ .

**Exercise**

**McCallum, page 903, Exercise 20.** Write a triple integral in spherical coordinates giving the volume of a sphere of radius  $K$  centered at the origin. Use the order  $d\theta d\rho d\phi$ .

Evaluate the triple integral to show the volume of the sphere of radius  $K$  is  $\frac{4}{3}\pi K^3$ .

GROUPWORK
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**McCallum, page 903, Exercise 24-25.**

Use (a) Cartesian (b) Cylindrical (c) Spherical coordinates to write down the limits of integration

for  $\int_W dV$  for the following figures.

**24.** One-eighth of the sphere with unit radius centered at the origin (occupying the positive  $x$ ,  $y$  and  $z$  quadrants)

**25.** The shape formed by a cone with  $90^\circ$  vertex at the origin topped by the sphere of radius 1 centered at the origin. (Sort of looks like an ice-cream cone.)