# Multivariable Calculus

Math 212 §2 Fall 2014 Fowler 309 MWF 11:45am - 12:40pm

http://faculty.oxy.edu/ron/math/212/14/

## *Worksheet 17*

**TITLE** Constrained Multivariable Optimization (Using Lagrange Multipliers)
**CURRENT READING** McCallum, Section 15.3
**HW #7 (DUE THURSDAY 10/16/14 5PM)**
McCallum, *Section 15.3*: 2, 5, 8, 14, 18, 21, 31, 34, 44*.
McCallum, *Chapter 15 Review*: 12, 23, 24, 25, 26, 41, 44*.

**SUMMARY**
This worksheet discusses the concept of optimizing a multivariable objective function $f(x, y)$ subject to a multivariable constraint function $g(x, y) = c$. This is a classic problem that is often solved by a technique called using Lagrange multipliers.

## Constrained Multivariable Optimization

Oftentimes we want to optimize (find the maximum/minimum of a particular function, called the **objective function** subject to a specific set of conditions, which is called the **constraint**.

EXAMPLE

**McCallum, page 850, Example 1**.
Find the maximum and minimum values of $x + y$ on the circle $x^2 + y^2 = 1$.

## Method of Lagrange Multipliers

A smooth objective function $f(\vec{x})$ has a maximum or minimum subject to a smooth constraint $g(\vec{x}) = c$ at a point $\vec{x}_0$ then either

The point $\vec{x}_0$ satisfies the equations $\vec{\nabla} f = \lambda \vec{\nabla} g$ and $g = c$

OR $\vec{x}_0$ is an endpoint of the constraint $g$

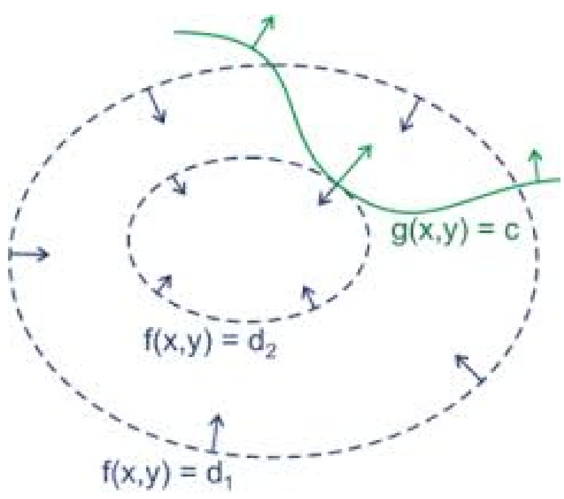
OR $\vec{\nabla} f(\vec{x}_0) = \vec{0}$

To find $\vec{x}_0$ compare values of the objective function $f$ at the points satisfying each of the above conditions. The number $\lambda$ is called the **Lagrange multiplier**.

EXAMPLE

**McCallum, page 850, Example 1**.
Let's find the maximum and minimum values of $x + y$ on the circle $x^2 + y^2 = 1$ using Lagrange Multipliers.

## Understanding Why Lagrange Multipliers Work




The reason why the method of Lagrange Multipliers works is related to the meaning of the gradient. We know that the gradient points in the direction of greatest increase of a function $f(x, y)$ and is always orthogonal to level sets of $f$.

The maximum and minimum value of $f(x, y)$ subject to the constraint $g(x, y) = c$ means that you are looking for a location where a level set of $f(x, y)$ is exactly tangential to the specific level set of $g(x, y) = c$. At this point grad $f$ will be parallel to grad $g$, as depicted in the Figure above.

 **Algorithm for Multivariable Optimization**

Given a function $f(\vec{x})$ defined inside and on a region $R$ in $\mathbb{R}^n$, compare the values of $f$ at the following points:

(a) Critical points of $f$ in the interior of $R$, where $\vec{\nabla} f = \vec{0}$

(b) Points on the Boundary of $R$

1. EITHER: Find a parametric representation $\vec{g}$ for the boundary of $R$, in which case we have a new optimization problem with the composite function $\vec{f}(\vec{g})$ defined on a set of one lower dimension,
2. OR: Use the Lagrange multiplier method by solving the system $\vec{\nabla} f = \sum_{k=1}^{n} \lambda_k \vec{\nabla} g_k$ where $g_k$ are the functions representing the $n$ constraints.

**Exercise**

**McCallum, page 852, Example 2**.

Find the maximum and minimum values of $f(x, y) = (x-1)^2 + (y-2)^2$ subject to the constraint $x^2 + y^2 \leq 45$

## Interpreting The Meaning of $\lambda$

The value of the Lagrange Multiplier has an actual physical meaning, it is the rate of change of the optimum value of the objective function $f(x, y)$ with respect to the increase in the value of the constraint $c$, where the constraint function was $g(x, y) = c$.
We can show this by the Chain Rule if we consider the optimum point of $f(x, y)$ under the constraint $g(x, y) = c$ to be the point $(x_0, y_0)$ where $x_0$ and $y_0$ are functions of $c$ so that $f(x_0(c), y_0(c))$ is the optimal value and $g(x_0(c), y_0(c)) = c$ is the constraint.

$$\begin{aligned}
\frac{df}{dc} &= \frac{\partial f}{\partial x}\frac{dx_0}{dc} + \frac{\partial f}{\partial y}\frac{dy_0}{dc} \\
&= \left(\lambda\frac{\partial g}{\partial x}\right)\frac{dx_0}{dc} + \left(\lambda\frac{\partial g}{\partial y}\right)\frac{dy_0}{dc} \\
&= \lambda\left(\frac{\partial g}{\partial x}\frac{dx_0}{dc} + \lambda\frac{\partial g}{\partial y}\frac{dy_0}{dc}\right) \\
&= \lambda\frac{dg}{dc} \\
&= \lambda
\end{aligned}$$

GROUPWORK

Adapted from **McCallum, page 856, Exercise 20.** Consider the contours of $f$ in the figure.
**(a)** Does $f$ have a maximum value subject to the linear constraint function $g(x, y) = c$ for $x \geq 0, y \geq 0$? If so, approximately where is it and what is its value?
**(b)** Does $f$ have a minimum value subject to the constraint? If so, approximately where is it and what is its value?
**(c)** Considering that $g(x, y) = c$ is linear in $x$ and $y$ what is the sign of $\lambda$? (In what direction does $g$ increase as $c$ increases?)