Multivariable Calculus

Worksheet 14

TITLE Differentiability of a Multivariable Function $f(x, y)$
CURRENT READING McCallum, Section 14.8
HW #6 (DUE WEDNESDAY 10/08/14)
McCallum, *Section 14.6*: 4, 11, 12, 26, 34, 35, 47*.
McCallum, *Section 14.7*: 6, 7, 8, 12, 19, 24, 30, 31,41*.
McCallum, *Section 14.8*: 3, 12, 19*.
McCallum, *Chapter 14*: 2, 14, 35, 45, 64*

SUMMARY
This worksheet discusses the concept of differentiability of multivariable functions of the form $f(x, y)$.

RECALL: differentiability
A function $f(x)$ is said to be differentiable at the point $x = a$ if the following limit exists

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

A function is differentiable at a point $x = a$ if and only if it is **locally linear** at that point.

THEOREM: differentiability implies continuity
If a function is differentiable at a point, then it is continuous at that point.

This statement is true if the function whether you are talking about single-variable functions like $y = f(x)$ or multivariable functions like $z = f(x, y)$!

The Differentiability of $f(x, y)$

Given a function $z = f(x, y)$ with partial derivatives f_x and f_y which are continuous at every point on a disk centered around (a, b), then f is differentiable at (a, b).

$$f_x \text{ and } f_y \text{ continuous at } (a, b) \Rightarrow f(x, y) \text{ differentiable at } (a, b)$$

NOTE: The existence of partial derivatives at appoint does NOT guarantee differentiability at that point! You need **continuity of those partial derivatives** to imply differentiability.

Exercise
McCallum, page 819, Example 5.
Show that the function $g(x, y) = \ln(x^2 + y^2)$ is differentiable everywhere in its domain.

`QUESTION:` What is the domain of $g(x, y) = \ln(x^2 + y^2)$?

Local Linearity Of A Multivariable Function $f(x, y)$ Implies Diferentiability

DEFINITION: differentiability at a point

The function $f(x, y)$ is said to be **differentiable at the point** (a, b) if there exists a linear function

$$L(x, y) = f(a, b) + p(x - a) + q(y - b)$$

such that the error $E(x, y)$ is defined so that $f(x, y) = L(x, y) + E(x, y)$where if $h = x - a$, $k = y - b$ then the relative error $E(a + h, b + k)/\sqrt{h^2 + k^2}$ satisfies the relationship

$$\lim_{\substack{h \to 0 \\ k \to 0}} \frac{E(a + h, b + k)}{\sqrt{h^2 + k^2}} = 0$$

We say $f(x, y)$ is **differentiable on a region** $\mathcal{R}$ if it is differentiable at each point of $\mathcal{R}$.
$L(x, y)$ is called the **local linearization** of $f(x, y)$ near (a, b).
$E(x, y)$ is a measure of the error being made at any point on the surface $z = f(x, y)$ being approximated by a tangent plane.

Informal Definition Of Differentiability At A Point

The formal definition of local linearity of a multivariable function given above basically means that if a surface $z = f(x, y)$ can be approximated very well by a plane at a particular point when one zooms in one the point then the function is differentiable at that point. We quantify "approximated very well" by showing that the limit of the error between the surface and the local linearization goes to zero when you get closer and closer to the point in question.

EXAMPLE

Adapted from **McCallum, page 819, Exercise 15**.
Consider

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

(a) Show that $f(x, y)$ is continuous at $(0, 0)$ by considering $x(t) = at$, $y(t) = bt$ where a and b are not zero at the same time and take the limit as t goes to zero of $g(t) = f(x(t), y(t)$,
(b) Show that $f(x, y)$ differentiable at $(x, y) \neq (0, 0)$.
(c) Show that $f_x(0, 0)$ and $f_y(0, 0)$ exist by using the limit definition of the partial derivative.
(d) Show that $f(x, y)$ differentiable at $(0, 0)$ by showing it is not locally linear at the origin.