Worksheet 9

TITLE The Directional Derivative and the Gradient Vector
CURRENT READING McCallum, Section 14.4 and 14.5.
HW #4 (DUE WED 09/24/14)
McCallum, *Section 12.6*: 24, 28, 35, 40, 52.
Section 14.1: 10, 11, 12, 13, 17, 18, 25,26, 36, 37, 41, 48, 49.
Section 14.2: 8, 9, 14, 24, 25, 30, 34, 36, 39, 51, 52, 65*.

SUMMARY
This worksheet discusses the concepts of the directional derivative, i.e. the rate of change of a multivariable function in the direction of a given unit vector, as well as the gradient vector $\vec{\nabla} f$.

The Directional Derivative

WARM-UP

Suppose we have a point P at (a,b) and we move a distance h in the direction $\hat{u} = u_1\vec{i} + u_2\vec{j}$ to reach point Q. What are the coordinates of the point Q? DRAW A PICTURE!

`QUESTION` Suppose $z = f(x, y)$ was a function with P and Q in its domain how could we estimate the rate of change of f from P to Q?

`ANSWER`

DEFINITION: **directional derivative**

The rate of change of the function $f(x, y)$ in the direction of the unit vector $\vec{u} = u_1\hat{i} + u_2\hat{j}$ at the point (a, b) is given by

$$f_{\vec{u}}(a, b) = \lim_{h\to 0} \frac{f(a + u_1h, b + u_2h) - f(a, b)}{h} \tag{1}$$

The directional derivative given in Equation (1) is sometimes denoted $D_{\vec{u}}(f)$.

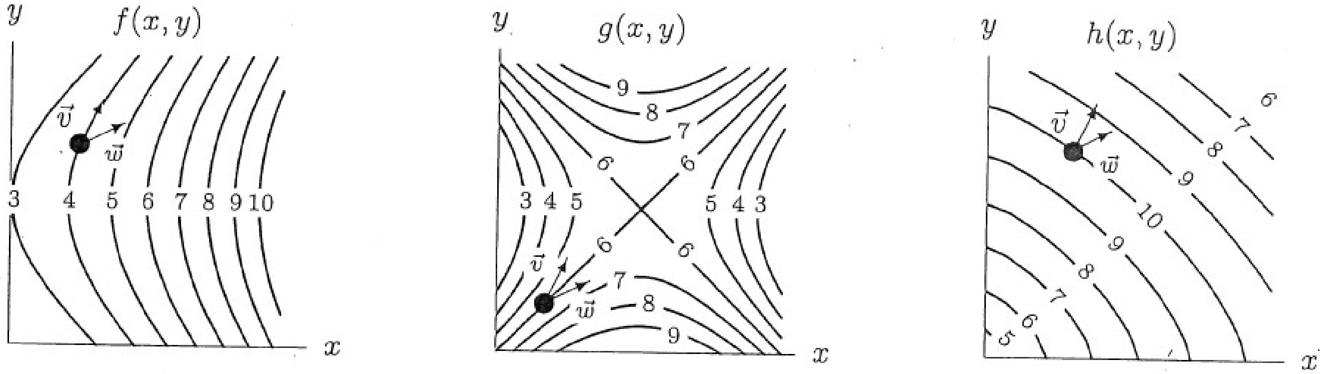
NOTE:

- If $\vec{u} = \hat{i}$ (i.e. $u_1 = 1$ and $u_2 = 0$) then the equation in (1) would just represent the partial derivative $f_x(a, b)$, i.e. the rate of change in the positive x direction.
- If $\vec{u} = \hat{j}$ (i.e. $u_1 = 0$ and $u_2 = 1$) then the equation in (1) would just represent the partial derivative $f_y(a, b)$, i.e. the rate of change in the positive y direction.
- The directional derivative is a **SCALAR** quantity!

Understanding the Directional Derivative Graphically

GroupWork

Consider the following contour diagrams for the functions $f(x,y)$, $g(x,y)$ and $h(x,y)$. Determine whether the directional derivative at the indicated point is **positive**, *negative* or zero in the direction of the vector $\vec{v} = \vec{i} + 2\vec{j}$ and in the direction of $\vec{w} = 2\vec{i} + \vec{j}$.



Computing Directional Derivatives

We already know how to compute f_x and f_y which are directional derivatives in the $\vec{i}$ direction and $\vec{j}$ direction, respectively.

Since every vector $\vec{u}$ in $\mathbb{R}^2$ can be written as a **linear combination** of a vector in the $\vec{i}$ direction and a vector in the $\vec{j}$ direction it should not be surprising that every directional derivative can be written as a linear combination of the partial derivatives of $f(x,y)$.

In other words, the directional derivative of $f(x,y)$ at (a,b) in the direction $\vec{u} = u_1\vec{i} + u_2\vec{j}$ can be written

$$f_{\vec{u}}(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2 = (f_x(a,b)\vec{i} + f_y(a,b)\vec{j}) \cdot (u_1\vec{i} + u_2\vec{j}) \qquad (2)$$

PROOF

Let's prove the result in (2), i.e. every directional derivative $f_{\vec{u}}(a,b)$ is the dot product of the vector of partial derivatives $f_x(a,b)\vec{i} + f_y(a,b)\vec{j}$ with a direction vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$.

Exercise

McCallum, Page 781, Example 3. Calculate the directional derivative of $f(x,y) = x^2 + y^2$ at (1,0) in the direction $\vec{i} + \vec{j}$.

The Gradient Vector

The result in (2) we proved is VERY useful because the vector composed of the partial derivatives of $f(x, y)$, i.e. $f_x(a, b)\vec{i} + f_y(a, b)\vec{j}$ is known as the **gradient vector** of $f(x, y)$ at the point (a, b). The gradient vector is sometimes written as $\vec{\nabla} f(a, b)$ or $\text{grad} f(a, b)$

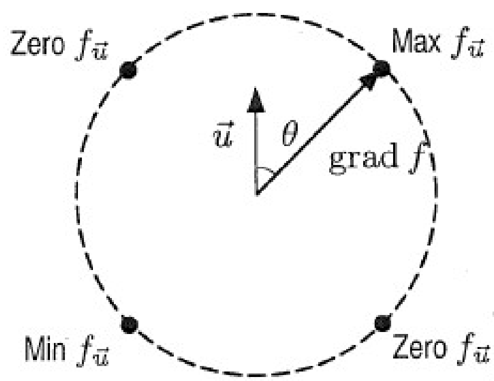
This means that we can re-write (2) to produce the formula we will use to compute directional derivatives from now on.

$$f_{\vec{u}}(a, b) = \vec{\nabla} f(a, b) \cdot \vec{u} \text{ or } \quad f_{\vec{u}}(a, b) = \text{grad} f(a, b) \cdot \vec{u} \tag{3}$$

Interpreting The Meaning Of The Gradient Vector

The directional derivative $f_{\vec{u}}(a, b)$ equals the magnitude of the gradient vector $||\vec{\nabla} f(a, b)||$ times cosine of the angle θ between $\vec{u}$ and the direction the gradient vector points in (by definition of the dot product)!

$$f_{\vec{u}}(a, b) = \vec{\nabla} f(a, b) \cdot \vec{u} = ||\vec{\nabla} f(a, b)|| ||\vec{u}|| \cos\theta = ||\vec{\nabla} f(a, b)|| \cos\theta \tag{4}$$



Values of the directional derivative at different angles to the gradient

Thus we can see from Equation (4) and the figure on this page that the rate of change of f in the direction $\vec{u}$ will be the largest when the angle between $\vec{\nabla} f(a, b)$ and $\vec{u}$ is ________________________________.

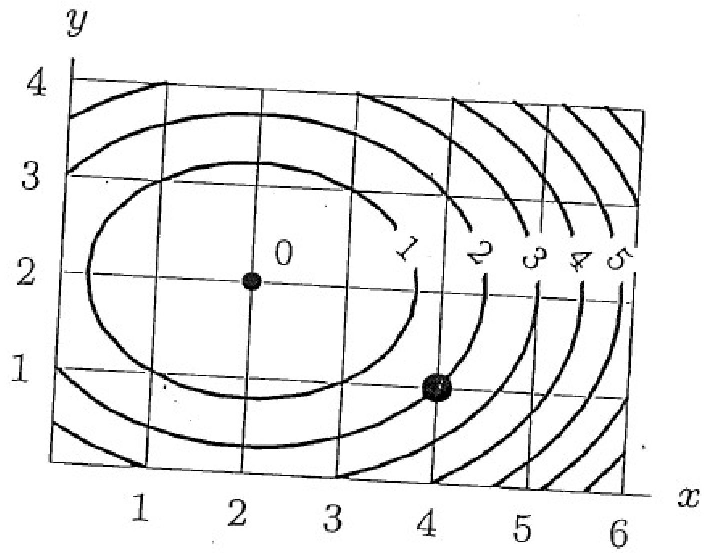
The Gradient Vector Points In The Direction Of Maximum Increase Of $f(x, y)$

This means that the gradient vector $\text{grad} f(a, b)$ always points in **the direction in which the functions changing the most**, also known as "the direction of maximum rate of increase of f at the point (a, b)."

The Gradient Vector Is (Always) Perpendicular To The Contours Of $f(x, y)$

The direction in which the function f will have the smallest rate of change in magnitude (i.e no change or the function will be *constant*) will occur when the angle between $\vec{u}$ and the gradient vector is ________________________________.

By definition, contours or level sets are curves along which the function f is constant, i.e. does not change, so since the gradient vector points in the direction of maximum change, it is always orthogonal to the direction of zero change. Thus the gradient vector always points in a direction perpendicular to the contours of f!



GROUPWORK

Adapted from McCallum, page 786, Problems 56-60. Use the figure above to calculate estimates of

(a) $f_{\vec{u}}(4,1)$ where $\vec{u} = \vec{i}$

(b) $f_{\vec{u}}(4,1)$ where $\vec{u} = \vec{j}$

(c) $f_{\vec{u}}(4,1)$ where $\vec{u} = (\vec{i} - \vec{j})/\sqrt{2}$

(d) $f_{\vec{u}}(4,1)$ where $\vec{u} = (-\vec{i} + \vec{j})/\sqrt{2}$

(e) $f_{\vec{u}}(4,1)$ where $\vec{u} = (-2\vec{i} + \vec{j})/\sqrt{5}$