Exam 1: Multivariable Calculus

Math 212 Fall 2014 | Friday November 7
Prof. Ron Buckmire | 11:45am-12:40pm

Name: BUCKMIRE

Directions:

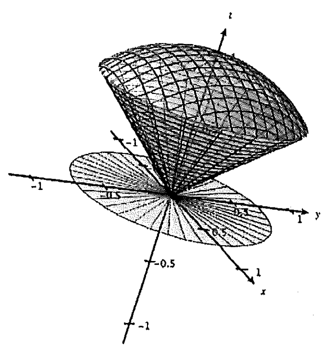
Read *all* problems first before answering any of them. This tests consists of four (4) problems (and a BONUS problem) on six (6) pages.

The topic of the problem is **in bold**, the number of points each problem is worth is in *italics* and the kind of skills required to solve each problem are in ALL CAPS.

This is a 55-minute, limited notes, closed book, test. **No calculators or electronic devices may be used.** Your notes are limited to one side of an 8.5" by 11" piece of paper, which must be handed in with your exam.

You must show all relevant work to support your answers. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your "scratch work."

Questions Policy: FEEL FREE TO ASK CLARIFICATION QUESTIONS AT ANY TIME!

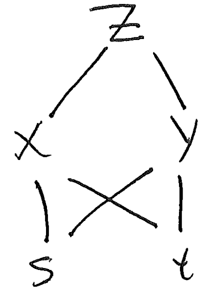


No.	Score	Maximum
1		30
2		20
3		20
4		30
BONUS		10
Total		**100**

1. (*30 points.*) VISUAL, ANALYTIC, COMPUTATIONAL. **Chain Rule, Partial Derivatives.** Let $x = s + t$ and $y = s - t$. Suppose $z = f(x, y)$ is a differentiable function of x and y, the goal of this question is to show that

$$\left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 = \frac{\partial f}{\partial s}\frac{\partial f}{\partial t}$$

1(a) (*6 points.*) Write down a tree diagram showing the relationships between z, x, y, s and t.



1(b) (*16 points.*) Knowing that $z = f(x(s,t), y(s,t))$ use the Chain Rule to obtain expressions for $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = f_x \cdot 1 + f_y \cdot 1$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = f_x \cdot 1 + f_y \cdot -1$$

$x_s = 1 \quad y_s = 1$

$x_t = 1 \quad y_t = -1$

1(c) (*8 points.*) Obtain an algebraic expression linking f_x and f_y with f_s and f_t, i.e. show that $f_x^2 - f_y^2 = f_s f_t$.

$$f_s \cdot f_t = (f_x + f_y)(f_x - f_y)$$
$$= f_x^2 - f_y^2$$

2. (*20 points.*) VERBAL, ANALYTIC, COMPUTATIONAL. **Unconstrained Multivariable Optimization, Extreme Value Theorem, Repeated Partial Differentiation.** Consider the function $f(x, y) = x^3 - xy - y^2 + y$.

2(a) (*16 points.*) Find the location of critical points of $f(x, y) = x^3 - xy - y^2 + y$ and use the Second Derivative Test to classify these points as local extrema or saddle points.

Critical points at $\vec{\nabla} f = \vec{0}$

$$f_x = 3x^2 - y = 0$$
$$f_y = -x - 2y + 1 = 0$$

$$y = 3x^2 \Rightarrow y = 3(1-2y)^2$$
$$x = 1 - 2y \qquad y = 3[1 + 4y^2 - 4y]$$
$$12y^2 - 12y + 3 - y = 0$$
$$12y^2 - 13y + 3 = 0$$
$$y = \frac{13 \pm \sqrt{(13)^2 - 4 \cdot 12 \cdot 3}}{2 \cdot 12} = \frac{13 \pm \sqrt{169 - 144}}{24} = \frac{13 \pm \sqrt{25}}{24} = \frac{13 \pm 5}{24} = \frac{18}{24} \text{ or } \frac{8}{24} = \frac{3}{4} \text{ or } \frac{1}{3}$$

$$y = \frac{1-x}{2}$$
$$\frac{1-x}{2} = 3x^2$$
$$1 - x = 6x^2$$
$$6x^2 + x - 1 = 0$$
$$x = \frac{-1 \pm \sqrt{1^2 - 4(-1)(6)}}{2 \cdot 6} = \frac{-1 \pm 5}{12} = -\frac{6}{12} \text{ or } \frac{4}{12}$$

When $y = 3/4$
$$x = 1 - 2\left(\tfrac{3}{4}\right) = -\tfrac{1}{2}$$
When $y = 1/3$
$$x = 1 - 2\left(\tfrac{1}{3}\right) = \tfrac{1}{3}$$

$(-1/2, 3/4)$ is a local max
$(1/3, 1/3)$ is a saddle

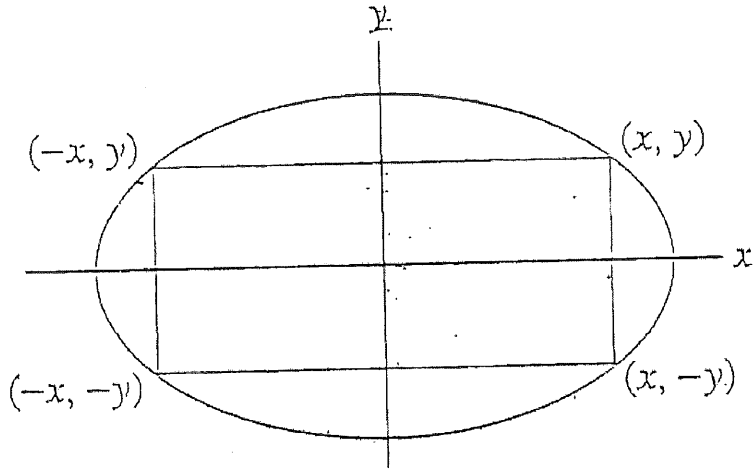
$$f_{xx} = 6x$$
$$f_{yy} = -2$$
$$f_{xy} = -1$$

$$D = f_{xx} f_{yy} - f_{xy}^2$$
$$D(-1/2, 3/4) = 6 \cdot -\tfrac{1}{2} \cdot -2 - (-1)^2 = 6 - 1 = 5 > 0$$
$$D(1/3, 1/3) = 6 \cdot \tfrac{1}{3} \cdot -2 - (-1)^2 = -5 < 0$$

2(b) (*4 points.*) Are the extrema you found in **2(a)** global extrema for $f(x, y)$? EXPLAIN YOUR ANSWER!

These are not global extrema because (a) one is a saddle
(b) $\lim_{x \to \infty} f = +\infty$
$\lim_{x \to \infty} f = -\infty$

3. (*20 points.*) ANALYTIC, VISUAL, VERBAL. **Constrained Multivariable Optimization, Lagrange Multipliers.**



Use Lagrange Multipliers to find the maximum area of the rectangle which is inscribed within the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ shown in the figure above.

$f = (2x)(2y) = 4xy$

$g = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

$f_x = 4y \qquad g_x = \frac{2x}{a^2}$

$f_y = 4x \qquad g_y = \frac{2y}{b^2}$

$\vec{\nabla} f = \lambda \vec{\nabla} g$

$4y = \lambda \frac{2x}{a^2}$

$4x = \lambda \frac{2y}{b^2}$

$4xy = \lambda \frac{2x^2}{a^2}$

$4xy = \lambda \frac{2y^2}{b^2}$

$\lambda \frac{x^2}{a^2} = \lambda \frac{y^2}{b^2}$

$\frac{x^2}{a^2} = \frac{y^2}{b^2} \Rightarrow \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$2x^2 = a^2$

$x^2 = \frac{a^2}{2} \Rightarrow \frac{y^2}{b^2} = \frac{1}{2}$

$y^2 = \frac{b^2}{2}$

$x = \frac{a}{\sqrt{2}}, \; y = \frac{b}{\sqrt{2}}$

$f = 4xy = 4 \frac{a}{\sqrt{2}} \cdot \frac{b}{\sqrt{2}}$

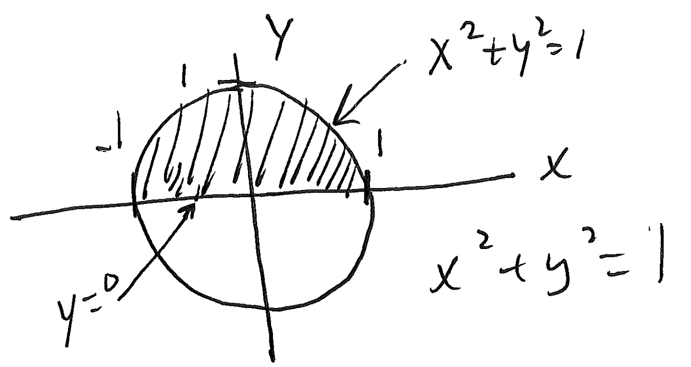
$= 2ab$

Maximum area of Rectangle is $2ab$.

4. (*30 points.*) ANALYTIC, VISUAL, COMPUTATIONAL. **Polar Coordinates, Iterated Integration, Multiple Integration.** Consider the following integral $\mathcal{I}$

$$\mathcal{I} = \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y \; dx \; dy$$

3(a) (*10 points.*) Draw a sketch of the region of integration in the xy-plane which is represented by the integral $\mathcal{I}$. Carefully label the boundaries of the region with the appropriate equations.



$$\mathcal{I} = \int_0^1 xy \Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy$$

$du = -2y\,dy$, $u = 1-y^2$, $y=0, u=1$, $y=1, u=0$

$$= \int_0^1 2y\sqrt{1-y^2}\,dy = \int_{-1}^{0} \sqrt{u}\,du = \int_0^1 u^{1/2}\,du$$

$$= \frac{2(1-y^2)^{3/2}}{3} \cdot \frac{1}{-1} \Big|_0^{y=1}$$

$$= \frac{2}{3}(1-1^2) - \frac{2}{3}\frac{(1)^{3/2}}{-1} = \boxed{2/3}$$

3(b) (*10 points.*) Obtain the value of $\mathcal{I}$ by evaluating a different integral than the one given above.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} y\,dy\,dx = \int_{-1}^{1} \frac{y^2}{2}\Big|_0^{\sqrt{1-x^2}} dx = \int_{-1}^{1} \frac{1}{2}(1-x^2)\,dx$$

$$= 2\int_0^1 \frac{1}{2}(1-x^2)\,dx = x - \frac{x^3}{3}\Big|_0^1$$

$$= 1 - \frac{1^3}{3} = \frac{2}{3}$$

3(c) (*10 points.*) Confirm the value of $\mathcal{I}$ obtained in **3(b)** by evaluating another integral (different from the one evaluated in **3(b)**).

$$\int_0^{\pi}\int_0^1 r\sin\theta\, r\,dr\,d\theta = \int_0^{\pi}\int_0^1 r^2\sin\theta\,dr\,d\theta = \int_0^{\pi} \frac{r^3}{3}\sin\theta\Big|_0^1 d\theta$$

$$= \int_0^{\pi} \frac{1}{3}\sin\theta\,d\theta = -\frac{1}{3}\cos\theta\Big|_0^{\pi}$$

$$= \frac{1}{3}\cos\theta\Big|_{\pi}^{0}$$

$$= \frac{1}{3}(\cos 0 - \cos\pi) = \frac{2}{3}$$

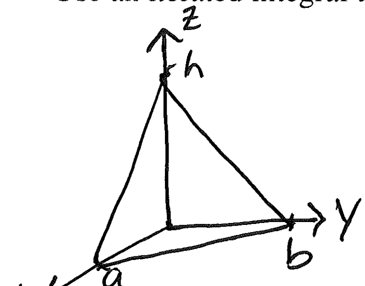
BONUS QUESTION. Triple Integration. (*10 points.*)

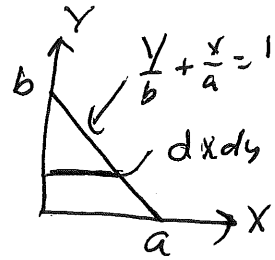
Use iterated integration to answer ONLY one of the following two problems. Make sure you draw clear diagrams indicating how you know the limits of integration in your iterated integrals.

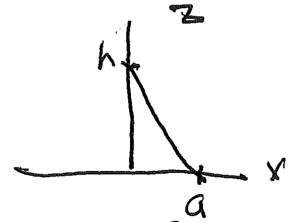
Use an iterated integral to compute the volume of a triangular pyramid of height h which has as its base a right-angled triangle with vertices at $(a,0,0)$, $(0,b,0)$ and $(0,0,0)$. The vertex of the pyramid is at $(0,0,h)$.

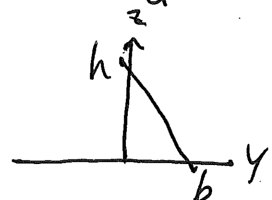
OR

Use an iterated integral to compute the volume of a circular cone with height h and radius R.









This is a plane above a triangular region

$z = h\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

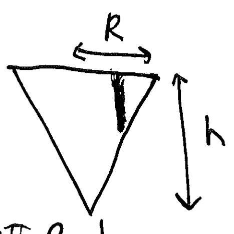
$$\int_0^b \int_0^{a(1-\frac{y}{b})} h\left(1-\frac{x}{a}-\frac{y}{b}\right) dx\,dy = h\int_0^b x - \frac{x^2}{2a} - \frac{xy}{b}\Big|_0^{(1-\frac{y}{b})a} dy$$

$$= h\int_0^b \left(\left(1-\frac{y}{b}\right)a - \left(1-\frac{y}{b}\right)^2\frac{a^2}{2a} - \left(1-\frac{y}{b}\right)\frac{y}{b}a\right) dy$$

$$= h\int_0^b a\left(1-\frac{y}{b}\right)\left[1 - \frac{1}{2}\left(1-\frac{y}{b}\right) - \frac{y}{b}\right] dy$$

$$= ha\int_0^b \frac{1}{2}\left(1-\frac{y}{b}\right)^2 dy = -\frac{ha}{2}\frac{1}{3}\left(1-\frac{y}{b}\right)^3\Big|_0^b$$

$$= \frac{1}{6}hab$$



$\frac{z}{r} = \frac{h}{R} \Rightarrow z = \frac{hr}{R}$

$$\int_0^{2\pi}\int_0^R\int_{\frac{hr}{R}}^h r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^R rh - \frac{hr^2}{R}\,dr\,d\theta$$

Cylindrical

$$= 2\pi h\int_0^R r - \frac{r^2}{R}\,dr$$

$$= 2\pi h\left(\frac{r^2}{2} - \frac{r^3}{3R}\right)\Big|_0^R$$

$$= 2\pi h\left(\frac{R^2}{2} - \frac{R^2}{3}\right)$$

$$= 2\pi h\frac{R^2}{6} = \frac{\pi R^2 h}{3}$$