SPRING 2004

4. (*20 points.*) **Multiple Integration.**

a. (*10 points*) Evaluate $\int\int_R ye^x \, dA$ where R is the first quadrant of the circle of radius 4 centered at the origin. (Sketch the region R).

b. (*10 points*) Consider $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} dz \, dy \, dx = \frac{1}{12}$. Re-compute this integral using a different triple integral which represents the same volume.

SPRING 2004

3. (*20 points.*) **Iterated Integration.**

a. (*10 points*) Evaluate $\displaystyle\int_{-3}^{0}\int_{0}^{2}\int_{-1}^{1} \cos(x+y+z) - xyz \; dx \; dz \; dy$

b. (*10 points*) Evaluate $\displaystyle\int_{1}^{2}\int_{0}^{\ln x} \frac{1}{x} \; dy \; dx$

5. (*20 points.*) **Constrained Multivariable Optimization, Lagrange Multipliers**
The "geometric mean" of n numbers is defined as $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 x_3 \ldots x_n}$. Suppose that $x_1, x_2, \ldots, x_n$ are positive numbers such that $\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots x_n = c$, where c is a constant.

a. (*10 points*) Find the maximum value of the geometric mean of n positive numbers given the constraint that their sum must be equal to a constant. [HINT: Consider f^n instead of f!]

b. (*10 points*) You can deduce from part (a) that the geometric mean of n numbers is always less than or equal to the arithmetic mean, that is:

$$\sqrt[n]{x_1 x_2 x_3 \ldots x_n} \leq \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

Under what conditions will the geometric mean be exactly equal to the arithmetic mean of those same n numbers?

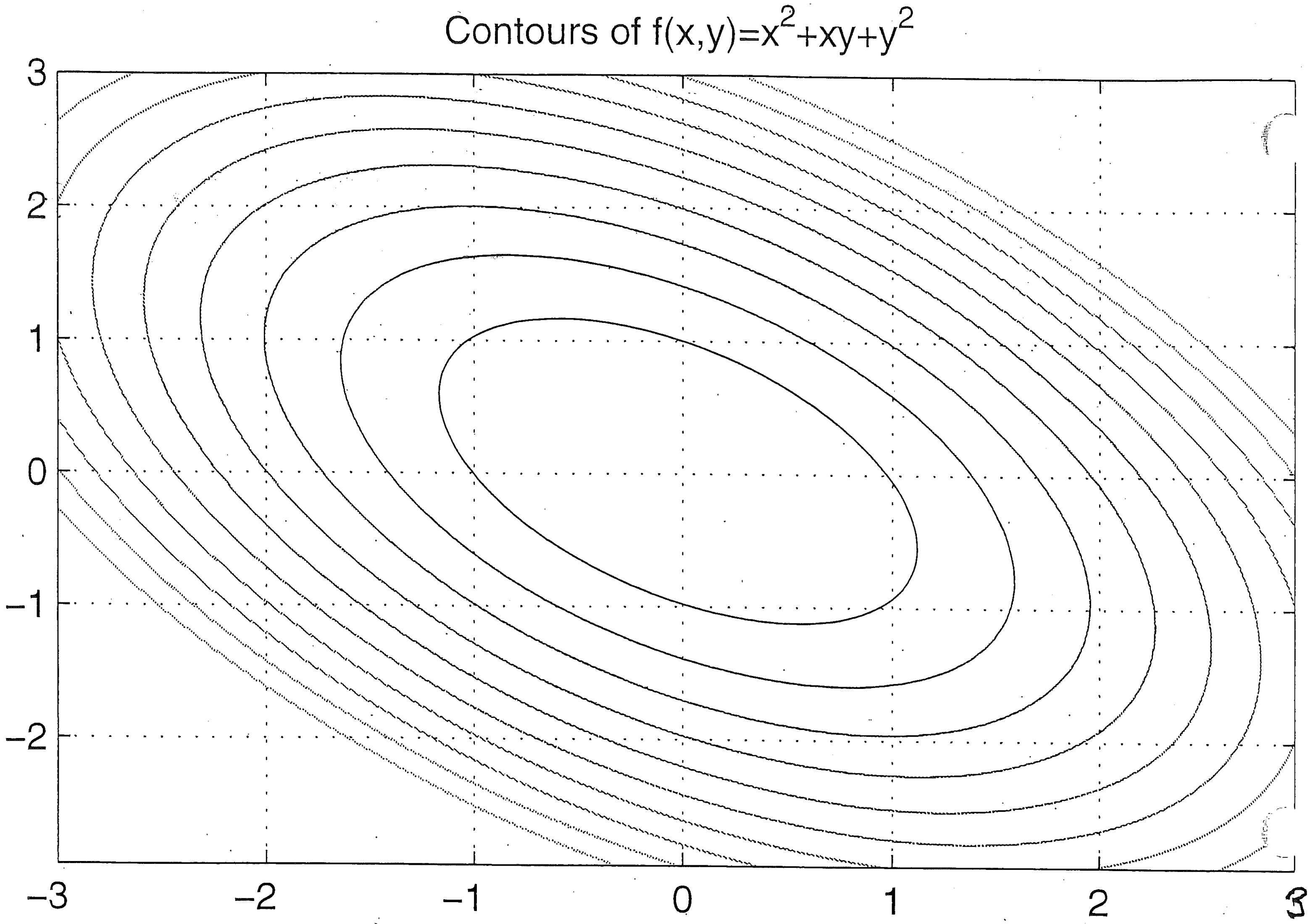
SPRING 2004

EXTRA CREDIT (*10 points.*) **Unconstrained Multivariable Optimization**
Consider $f(x, y) = x^4 + y^4 - 4xy + 1$.
a. (*5 points*) Find the three critical points of $f(x, y)$.

b. (*5 points*) Use the Second Derivative Test to classify each of the three critical points of $f(x, y)$.

SPRING 2004

SPRING 2006



(e) Using the picture alone, estimate the points at which the objective function $f(x, y)$ achieves a global minimum on the constraint set $g(x, y) = 0$ and the values of f there. **EXPLAIN YOUR ANSWER.**

SPRING 2006

(f) Use the Method of Lagrange Multipliers to obtain the minimum value of $f(x, y) = x^2 + xy + y^2$ on the constraint set $g(x, y) = x + y - 2 = 0$.

(g) How would the maximum and minimum on the constraint set change if the constraint set $g(x, y)$ were changed to $h(x, y) = x^2 + y^2 - 4$? Find the extrema of $f(x, y) = x^2 + xy + y^2$ subject to the constraint $h(x, y) = 0$ and EXPLAIN YOUR ANSWER.

FALL 2005

2. (*20 points.*) **Multiple Integration.**

The goal of this question is to evaluate $\int_0^\infty e^{-x^2}\,dx = \lim_{a\to\infty}\int_0^a e^{-x^2}\,dx$.

(a) (*10 points.*) Find $I(R) = \iint_{D_R} e^{-(x^2+y^2)}\,dx\,dy$ when D_R is $x^2+y^2 \le R^2$ (the interior of the circle of radius R centered at the origin). **HINT: pick a useful coordinate system!**

(b) (*5 points.*) Take your answer $I(R)$ to **(b)** and then let $R \to \infty$. What is $\lim_{R\to\infty}\iint_{D_R} e^{-(x^2+y^2)}\,dx\ \,dy$?

(c) (*5 points.*) Given that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \left[\int_{-\infty}^{\infty} e^{-x^2}\,dx\right]^2$ then what is the value of $\int_0^\infty e^{-x^2}\,dx$?

FALL 2005

4. (*20 points.*) **Iterated Integration.**

Consider the iterated integral for $V = \int_{-1}^{1}\int_{0}^{1}\int_{x^2}^{1} dz\ dy\ dx = \frac{4}{3}$

(a) (*12 points.*) Write down 3 (**three**) of the 5 (**five**) other possible triple iterated integrals which represent the exact same value V. **HINT: There is no dependence of z upon y)** DO NOT EVALUATE THESE INTEGRALS.

(b) (*8 points.*) Use any one of the iterated integrals you wrote down in part (a) to confirm the value of V.

SPRING 2006

2. Multivariable Chain Rule. *25 points.*

Consider the functions $u(x, y, z) = f(x - y, y - z, z - x)$. Our goal is to show that a function u with this form satisfies the following famous partial differential equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

(a) (*10 points.*) Consider a function $u = f(r, s, t)$ where $r = r(x, y, z)$, $s = s(x, y, z)$ and $t = t(x, y, z)$ are given. In other words, although u is a function of r, s and t, since each of these functions is a function of x, y and z one can consider u as a function of x, y and z. Use the Chain Rule to write down expressions for $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, and $\frac{\partial u}{\partial z}$. **[HINT: draw a "tree diagram" reflecting the relationships between the variables to assist you.]**

(b) (*15 points.*) Let $r = x - y$, $s = y - z$ and $t = z - x$. Use this information and your answer to (a) to show that $u(x, y, z) = f(x - y, y - z, z - x)$ satsfies the equation $u_x + u_y + u_z = 0$.

FALL 2005

5. (*20 points.*) **Constrained Multivariable Optimization, Lagrange Multipliers**
Recall the Cobb-Douglas function $P(L,K) = bL^{\alpha}K^{1-\alpha}$ where the total production P of a certain product depends on the amount of labor L used and the amount K of capital investment ($0 < \alpha < 1$ and $b > 0$.)
If the cost of a unit of labor is m and the cost of unit of capital is n, given that the production of the company is fixed at a level Q, what values of L and K will minimize the cost function $C(L,K) = mL + nK$?

a. (*10 points*) Write down the equations you need to solve simultaneously to find the answer to the question.

b. (*10 points*) Solve the equations to find the values of L and K which minimize the cost function $C(L,K)$. (**HINT: Eliminate the Lagrange Multiplier first**).