Test 1: Multivariable Calculus

Math 212
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9:30pm-10:30am

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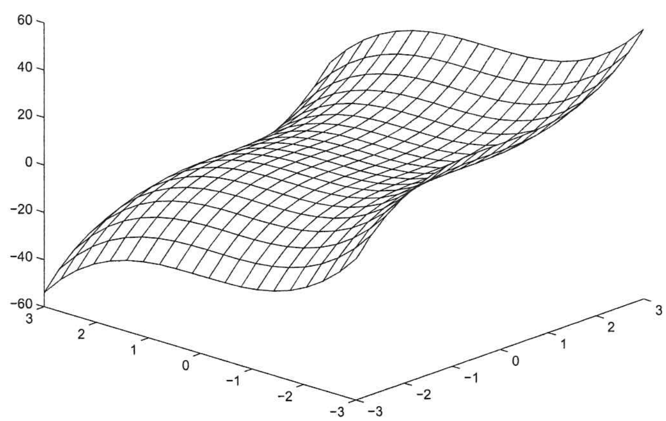
Directions: Read *all* problems first before answering any of them. Questions 2-4 are all related, but different. There are 7 pages in this test. This is a one hour, open-notes, open book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."



No.	Score	Maximum
1		30
2		20
3		20
4		30
BONUS		5
Total		**100**

1. Equation of Planes, Vector Operations. You are given the following three points in the plane:

$$A = (1, 2, 3) \qquad B = (2, 2, 5) \qquad C = (-1, 3, 4)$$

(a) (*6 points.*) Find the vector $\vec{v}$ which starts at A and points to B, and the vector $\vec{w}$ which starts at A and points to C.

$$\vec{v} = B - A = (2,2,5) - (1,2,3) = (1,0,2)$$

$$\vec{v} = \hat{\imath} + 2\hat{k}$$

$$\vec{w} = C - A = (-1,3,4) - (1,2,3) = (-2,1,1)$$

$$\vec{w} = -2\hat{\imath} + \hat{\jmath} + \hat{k}$$

(b) (*4 points.*) Find $\vec{v} \cdot \vec{w}$. Explain in complete sentences what this tells you about the angle between the two vectors and why.

$$\vec{v} \cdot \vec{w} = 1 \cdot (-2) + 0 \cdot 1 + 2 \cdot 1$$

$$= -2 + 0 + 2$$

$$= 0$$

$\vec{v}$ and $\vec{w}$ are orthogonal, i.e. the angle between them is 90°.

(c) (*10 points.*) Find $\vec{v} \times \vec{w}$. Explain in complete sentences what this tells you about the area of the triangle ABC.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 2 \\ -2 & 1 & 1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix}$$

$$= \hat{\imath}(-2) - \hat{\jmath}(1-(-2)) + \hat{k}\,1$$

$$= -2\hat{\imath} - 5\hat{\jmath} + \hat{k}$$

$$= (-2, -5, 1)$$

$$\text{Area of } \triangle ABC = \frac{1}{2}|\vec{v} \times \vec{w}| = \frac{1}{2}\sqrt{4+25+1} = \frac{1}{2}\sqrt{30}$$

(d) (*10 points.*) Find the equation of the plane that contains all three points A, B, and C.

We know $(-2,-5,1)$ is normal to the plane containing

$$-2(x-1) - 5(y-2) + 1(z-3) = 0$$

$$-2x + 2 - 5y + 10 + z - 3 = 0$$

$$-2x - 5y + z + 9 = 0$$

2. Level Sets, Vertical Slices. For the rest of the exam we will be considering the surface $z = f(x, y) = x^3 - y^3$.

(a) (*10 points.*) Identify which of the following graphs represents the level sets $f(x, y) = k$ or different vertical slices ($x = k$ or $y = k$) of the function for $k = -2, -1, 0, 1, 2$.

Figure 1

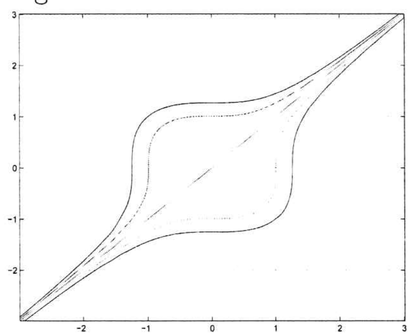


Figure 2

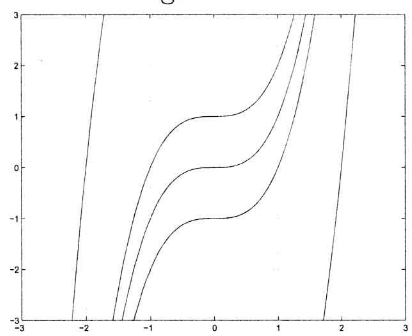
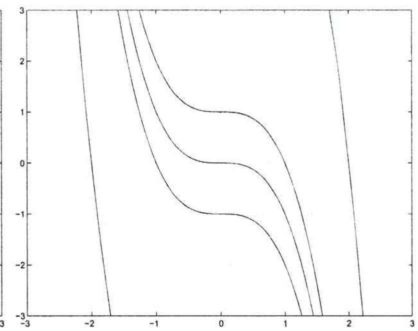


Figure 3



CLEARLY LABEL WHICH GRAPH REPRESENTS HOLDING WHICH VARIABLE CONSTANT AND FULLY EXPLAIN YOUR CHOICE BELOW.

Figure 1 is level sets $z = K$

Figure 2 is vertical slices $(y = K)$

Figure 3 is vertical slices $(x = K)$

The difference between Fig 2 and Fig 3 is Fig 2 looks like $z = x^3 - K$ while Fig 3 looks like $z = K - y^3$

(b) (*10 points.*) Explain how you can use the figures above to estimate that $f_{xy} = f_{yx} = 0$ at the origin (0,0). **What is another way you could show this result is true everywhere in the (x, y)-plane?**

$f = x^3 - y^3$

$f_x = 3x^2 \qquad f_{xy} = 0$

$f_y = -3y^2 \qquad f_{yx} = 0$

Fig 2: At the origin the rate of change of z with x near $x = 0$ is the same as one moves vertically, so $\frac{\Delta}{\Delta y}\left(\frac{\Delta z}{\Delta x}\right) = 0$. A similar argument applies to Fig. 1: $\frac{\Delta}{\Delta x}\left(\frac{\Delta z}{\Delta y}\right) = 0$.

3. Tangent Plane Approximation.

(a) (*10 points.*) Find the equation of the tangent plane to the surface $f(x, y) = x^3 - y^3$ at the point $(x, y) = (1, 2)$.

$f(1,2) = 1^3 - 2^3 = -7 \quad f_x = 3x^2 \quad f_y = -3y^2$

$f(x,y) \approx T(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$

$T(x,y) = -7 + 3(x-1) - 12(y-2)$

$T(x,y) = -7 + 3x - 3 - 12y + 24$

$T(x,y) = -12y + 3x + 14$

(b) (*10 points.*) Use this tangent plane approximation of the surface at this point to **estimate** the value of $(0.9)^3 - (1.99)^3$. [Note: I know the exact value is -7.151599. I'm looking for an estimate of this using the tangent plane approximation.]

$T(0.9, 1.99) = -7 + 3(0.9-1) - 12(1.99-2)$

$= -7 + 3(-0.1) - 12(-0.01)$

$= -7 - 0.3 + 0.12$

$= -7.3 + 0.12$

$= -7.18$

4. Gradient, Directional Derivative.

(a) (*15 points.*) The gradient of a function $f(x, y)$ evaluated at a point (x_0, y_0) is a vector pointing in the direction of the maximal rate of change of this function $f(x, y)$ at the point (x_0, y_0).

In what direction would you go from the point $(1, 2)$ to follow the maximal rate of change on $f(x, y) = x^3 - y^3$? What is the magnitude of this maximal rate of change?

$\vec{\nabla} f(x,y) = (3x^2, -3y^3)$

$\vec{\nabla} f(1,2) = (3, -12)$

If you go in the direction $3\hat{\imath} - 12\hat{\jmath}$ that is the direction of maximal rate of change of f.

$|\vec{\nabla} f(1,2)| = \sqrt{3^2 + 12^2} = \sqrt{9+144} = \sqrt{153}$

(b) (*10 points.*) What would the rate of change have been if you went in the direction $\vec{w} = \frac{4}{5}\vec{\imath} - \frac{3}{5}\vec{\jmath}$?

$$\frac{\partial f}{\partial \vec{\omega}} = \vec{\nabla} f \cdot \vec{\omega} = (3, -12) \cdot \left(\frac{4}{5}, \frac{-3}{5}\right)$$

$$= \frac{12}{5} + \frac{36}{5} = \frac{48}{5} = \boxed{9.6}$$

(c) (*5 points.*) What is a vector direction you can move in if you want the rate of change of $f(x, y) = x^3 - y^3$ at (1,2) to be zero?

The direction to move in would be such that $\nabla f(1,2) \cdot \vec{v} = 0$ where $\vec{v} = (v_1, v_2)$

$\vec{v} = (12, 3)$

$3v_1 - 12v_2 = 0$

BONUS QUESTION. Continuity, Set Theory. (*5 points.*)

Consider $g : \mathbb{R}^2 \to \mathbb{R}$ where $g(x,y) = \dfrac{f(x,y)}{x-y} = \dfrac{x^3 - y^3}{x-y}$. Describe the domain of the function $g(x,y)$. What kind of set (open, closed, et cetera) is it? Is the function $g(x,y)$ continuous on this domain? **EXPLAIN YOUR ANSWER THOROUGHLY, EXTRA CREDIT POINTS ARE HARD TO GET.**

The domain is $\{(x,y) \in \mathbb{R}^2 \setminus \{y = x\}\}$. All (x,y) pairs in $\mathbb{R}^2$ except where $x = y$.

This set is open. Every point in the set has a neighborhood which only contains points of the set.

The set is NOT CLOSED. The points along $y = x$ are all limit points of the set but are NOT in the set, thus the set does not

$g(x,y)$ __IS__ continuous everywhere in its domain. $g(x,y) = \dfrac{x^3 - y^3}{x - y} = \dfrac{(x-y)(x^2+xy+y^2)}{x-y}$

$g(x,y) = x^2 + xy + y^2$