Math 120 **CALCULUS 2** Spring 2014

Class 30: Wednesday April 23

Special Topic: Fourier Series, Part 2

`Warm-Up`

(a) Which of the following functions are even functions?

$f(x) = x^2 - x$ $\quad$ $h(t) = \cos(2t)$ $\quad$ $g(r) = r^3$ $\quad$ $b(y) = y^4 + 7.$

(b) Are all functions either even or odd?

DEFINITION: even and odd function

A function $f(x)$ is called **odd** if $f(-x) = -f(x)$ and it is called **even** if $f(-x) = f(x)$. Graphically, an even function is symmetric about the y-axis. An odd function is symmetric about the origin.

THEOREM: **Products of Odd and Even Functions**

The product of an even function and an odd function is an odd function. The product of two even functions is even and the product of two odd functions is also even.

THEOREM: **Integrals of Odd and Even Functions**

Suppose $E(x)$ is an even function and $O(x)$ is an odd function.

$$\int_{-a}^{a} E(x)\,dx = 2\int_0^a E(x)\,dx \text{ and } \int_{-a}^{a} O(x)\,dx = 0$$

General Formula To Find Fourier Series For Function With Period 2T On [-T,T]

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi}{T}t\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{T}t\right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{2T}\int_{-T}^{T} f(t)\,dt \\ a_k &= \frac{1}{T}\int_{-T}^{T} f(t)\,\cos\left(\frac{k\pi}{T}t\right)\,dt \\ b_k &= \frac{1}{T}\int_{-T}^{T} f(t)\,\sin\left(\frac{k\pi}{T}t\right)\,dt \end{aligned}$$

How can we use our knowledge about odd and even functions to obtain simpler formulas for the Fourier series representation of even and odd functions?

$\sin(x)$ is an ____________ function. $\cos(x)$ is an ____________ function.

If $E(x)$ is an even function, then $E(x)\cdot\sin(x)$ is ODD, while $E(x)\cdot\cos(x)$ is EVEN.

If $O(x)$ is an odd function, then $O(x)\cdot\sin(x)$ is EVEN, while $O(x)\cdot\cos(x)$ is ODD.

Fourier Series for Odd or Even Functions

If $g(t)$ is an **odd function** that is periodic on $[-T, T]$ with period $2T$ then its Fourier Series will only involve the **sine terms**, so $a_k = 0$ for all k. So,

$$b_k = \frac{2}{T}\int_0^T g(t)\sin\left(\frac{k\pi}{T}t\right)\,dt \qquad \text{for } k = 1, 2, 3, \ldots$$

If $f(t)$ is **an even function** with period $2T$ then its Fourier Series will only involve **the cosine terms**, so $b_k = 0$ for all k. So,

$$a_0 = \frac{1}{T}\int_0^T f(t)dt \text{ and } a_k = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{k\pi}{T}t\right)dt \qquad \text{for } k = 1, 2, 3, \ldots$$

$\boxed{\text{EXAMPLE}}$

$$f(x) = \begin{cases} x & \text{if } -1 \le x \le 0 \\ -x & \text{if } 0 < x < 1 \end{cases}$$

Sketch the graph of the function. It is sometimes called the **triangle wave**. What is the period?

Will the Fourier Series representation of this function involve cosine or sine or both?

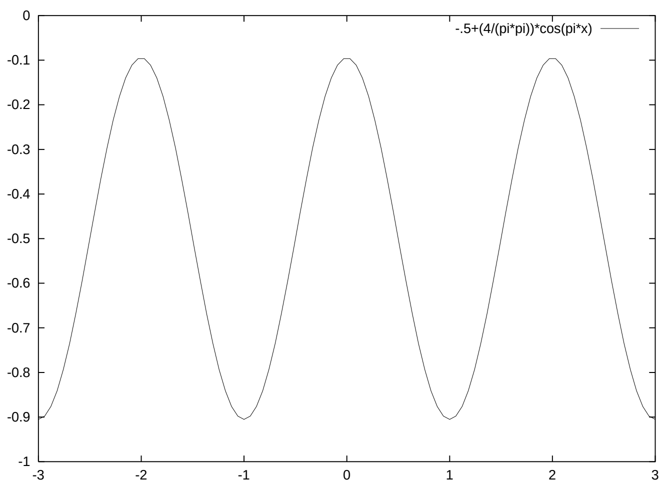
Find the first degree Fourier polynomial.

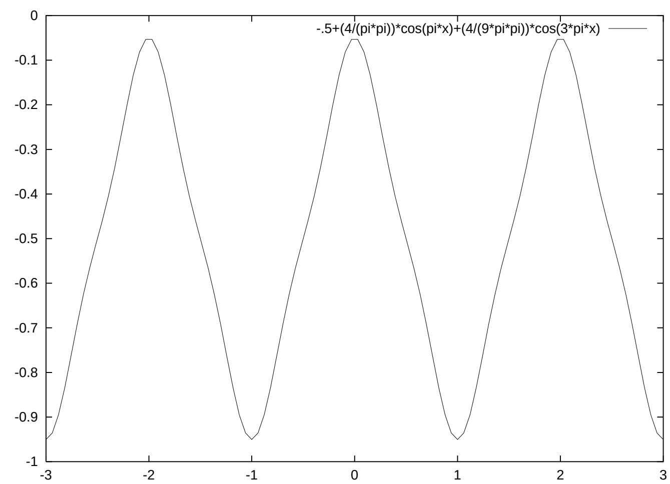
Exercise

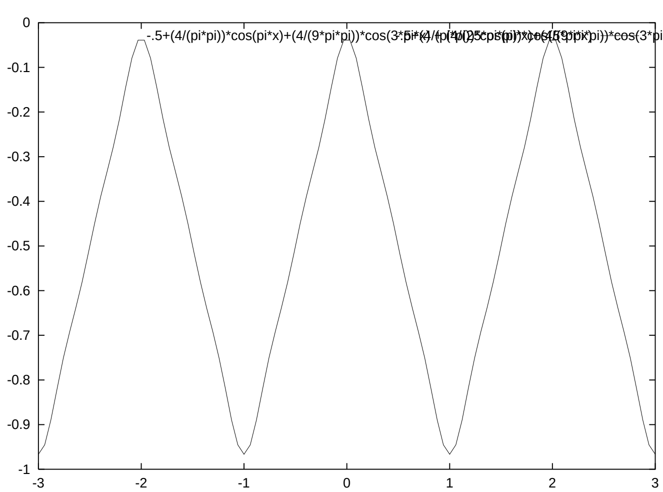
Find the general expression for the Fourier Series for our function $f(x)$ and write down the first few terms in the series.

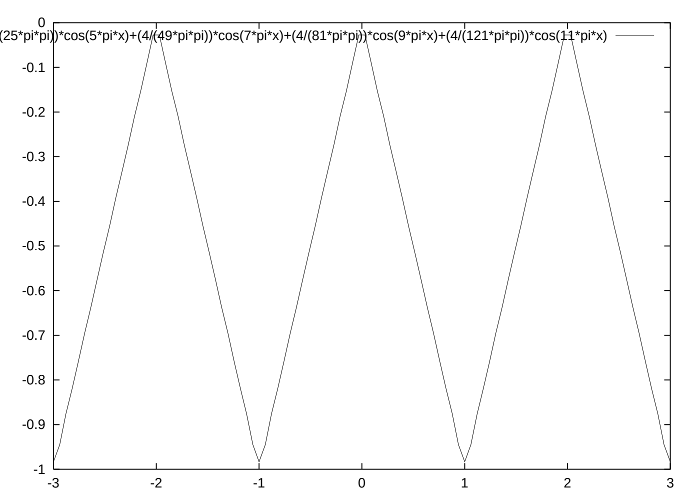
HINT: $\int x\cos(mx)dx = \frac{x}{m}\sin(mx) + \frac{1}{m^2}\cos(mx)$

The Fourier Series representation for the triangle wave looks like $-\frac{1}{2} + \sum_{k=1}^{\infty} \frac{2[1-(-1)^k]}{(k\pi)^2}\cos(k\pi x)$









DEFINITION: **Energy Spectrum of a periodic function**
The energy E of a periodic function f of period $2T$ is

$$E = \frac{1}{T}\int_{-T}^{T}[f(x)]^2\,dx = A_0^2 + A_1^2 + \dots$$

where $A_k = \sqrt{a_k^2 + b_k^2}$ and $A_0 = a_0\sqrt{2}$

The graph of A_k^2 versus k is called the **energy spectrum** of a periodic function and demonstrates how the energy of a periodic function is distributed among its harmonics. Typically, most of the energy of a periodic function is in the lower harmonics.

EXAMPLE
Let's find the energy spectrum of the Fourier series for the $f(x)$ we just computed.

Fourier Series versus Taylor Series

Taylor uses polynomials to approximate functions; Taylor polynomials are accurate **locally**. Fourier uses trigonometric functions to approximate **periodic** functions; Fourier polynomials are accurate **globally**.

We write $T_n(x)$ for the n^{th} degree Taylor polynomial.
Example: $T_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

We write $F_n(x)$ for the n^{th} degree Fourier "polynomial."
Example: $F_3(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x)$.

GroupWork
Compare and contrast Fourier series and Taylor series. What processes does one use to compute the coefficients? In what ways are they similar? In what ways are they different?