Calculus 2

Class 29: Monday April 21

Special Topic: Fourier Series, Part 1

`Warm-Up`

Write down an example of a periodic function that you know and sketch it below:

(Informal) DEFINITION: periodic

A function $f(t)$ is called **periodic** if it "repetitive," i.e., if its graph "repeats itself." In other words, the function produces the same output values in sequence, for a different set of inputs, in cyclic fashion.

The **period** of $f(t)$ is the "time it takes for the graph to repeat itself" (or, the time it takes to go through one cycle).

DEFINITION: **periodic function**

A function $f(t)$ is called **periodic** if for every t, $f(t+p) = f(t)$, where p is some non-zero constant number.

p is called the **period** of $f(t)$ (assuming it's the smallest possible such number which satisfies $f(t+p) = f(t)$).

GroupWork

Label each of the following as periodic or not periodic. If the function is periodic, find its period.

(a) $f(x) = \sin(x)$ (b) $g(t) = t^2$ (c) $f(x) = x^2 \sin(x)$

(d) $f(t) = t$ (e) $f(t) = 4$ (f) $h(x) = \cos(2x)$

(g) $f(x) = \begin{cases} 2 & \text{if } 2n \le x \le 2n+1 \\ 1 & \text{if } 2n+1 < x < 2n+2 \end{cases}$ where n is an integer

Pick which functions you think are periodic, sketch them below, and indicate their period

Fourier Series

In general, a Fourier Series is used to approximate a function $f(t)$ with period 2π

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kt) + \sum_{k=1}^{\infty} b_k \sin(kt)$$

where

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\ dt \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \cos(kt)\ dt \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \sin(kt)\ dt \end{aligned}$$

This usually involves a fair amount of integration to find explicit forms of the coefficients a_k and b_k. **NOTE:** a_0 is the average value of $f(x)$ on the interval $[-\pi, \pi]$.

EXAMPLE

Consider the following function, which is a famous signal called a **square wave**.

$$f(x) = \begin{cases} -1 & \text{if } -\pi \le x \le 0 \\ 1 & \text{if } 0 < x < \pi \end{cases}$$

1. Sketch the graph of $f(x)$ below between $-2\pi \le x \le 3\pi$.

2. Find the zeroth degree Fourier polynomial for $f(x)$.

2. Find the first degree Fourier polynomial for $f(x)$.

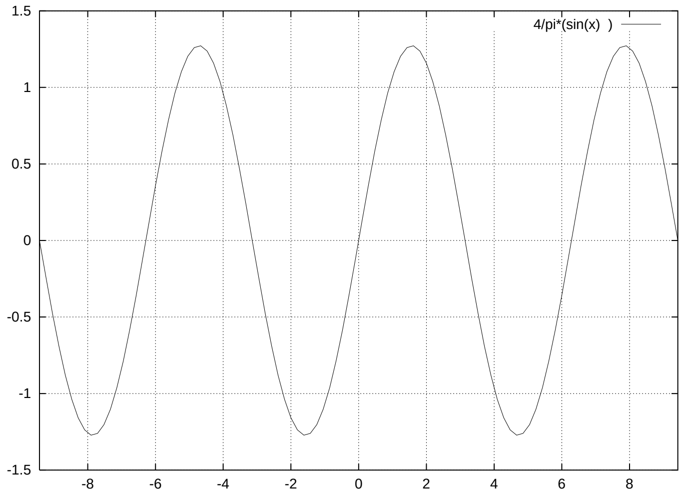
Exercise

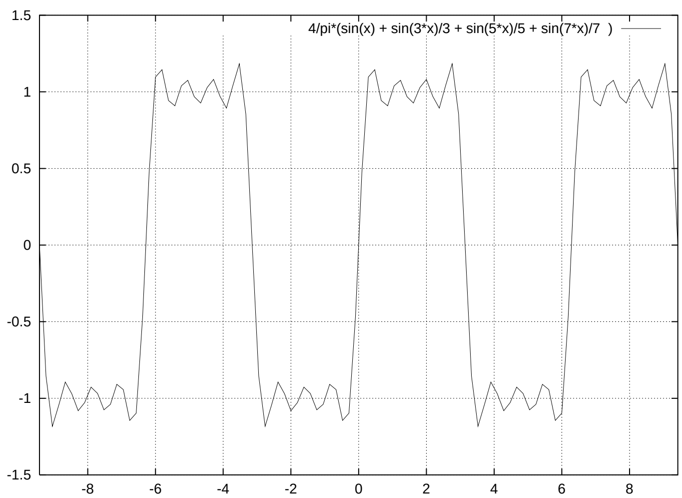
3. Show the general form of the Fourier polynomial for this $f(x)$ is $+\sum_{k=1}^{\infty} \frac{2}{\pi}[1-(-1)^k]\sin(kx)$.

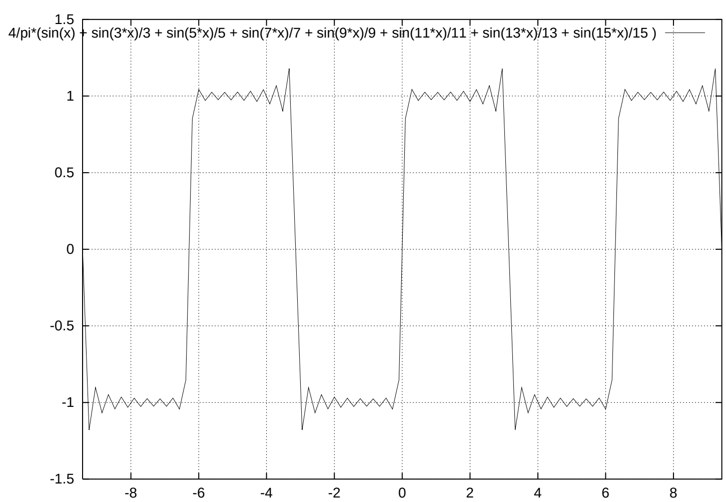
4. Write down the 7th degree Fourier polynomial approximation to the square wave.

5. For what values of x will the infinite series converge? What happens when you try absolute ratio test?

The Fourier polynomials $F_N(x) = \sum_{k=1}^{N} \frac{2}{\pi}[1-(-1)^k]\sin(kx)]$ is graphed below. The figures show $F_1(x)$, $F_7(x)$ and $F_{15}(x)$







What do you think the graph of $F_\infty(x)$ looks like?