CALCULUS 2

Class 24: Wednesday April 2

Power Series

Warm-Up

(a) What limit does the sequence $10^{0.1}$, $100^{0.01}$, $100000^{0.00001}, \ldots$ represent? Does this sequence have a limit? If so, what is it?

(b) What is the degree of the polynomial $p(x) = x^7 - x^3 + 4x^2 + 2x - 13$?

DEFINITION: **Power Series**

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

where x is variable and the constants c_n are known as the **coefficients** of the series.

If the power series converges to a value for certain values of x then we can consider it a function $f(x)$ and think of it as a polynomial with infinitely many terms or an infinite degree polynomial.

EXAMPLE

Suppose $c_n = 1$ for all n, then the power series looks like $\sum_{n=0}^{\infty} x^n$. Write out the first 5 terms. Does this series remind you of anything?

Recall, that we can use the Absolute Ratio Test to test for convergence of series. Let's do that.

Thus, we have shown that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{when } -1 < x < 1$$

Exercise

Stewart, page 465, Example 2. For what values of x does the power series $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$ converge?

THEOREM

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(i) The series converges only when $x = a$

(ii) The series converges for all x values

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and the series diverges if $|x-a| > R$

DEFINITION: **Radius of Convergence and Interval of Convergence**

The **radius of convergence** R of a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is defined to be

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$$

and the **interval of convergence** is either (i)$a-R < x < a+R$ or $(a-R, a+R)$; (ii) $a-R \le x < a+R$ or $[a-R, a+R)$; (iii) $a-R < x \le a+R$ or $(a-R, a+R]$; or (iv) $a-R \le x \le a+R$ or $[a-R, a+R]$ when the interval of convergence is not a single point (i.e. $R = 0$) or all x-values ($R = \infty$).

EXAMPLE

Stewart, page 465, Example 3. Find the domain of the Bessel function of order 0 defined by $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

Exercise

Stewart, page 467, Example 5. Find the radius of convergence and the interval of convergence of $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$.