Calculus 2

Class 21: Wednesday March 26

Introduction to Infinite Series

Warm-Up

(a) Can a sum of an infinite list of positive numbers be finite?

Summing an Infinite List of Numbers.

If someone gives you a list of numbers, even a long list of numbers (like 1000 of them), it is at least theoretically possible for you to use your calculator or computer to find the sum total of this list. Now, suppose someone gives you an *infinite* list of numbers, for example, the sequence $\left\{\frac{1}{k^k}\right\}_{k=1}^{\infty}$:

$$1, \qquad 1/2^2, \qquad 1/3^3, \qquad 1/4^4, \qquad 1/5^5, \qquad 1/6^6, \qquad \textit{etc.}$$

Is it possible to find the total? What could "find the total" mean if you are adding up an infinite list of numbers?

GroupWork

In small groups use your calculators to begin with the first number on the infinite list above, 1, and progressively add each successive number on the list, keeping track of the subtotals you get by placing them in the chart below, with seven places after the decimal.

n	n^{th} subtotal	
1	1	$= 1.0000000\ldots$
2	$1 + 1/2^2$	$=$
3	3^{rd} subtotal	$=$
4	4^{th} subtotal	$=$
5	5^{th} subtotal	$=$
6	6^{th} subtotal	$=$

What do you find happening to the subtotals? If this trend continues, what will be the first four digits of all the subtotals beyond those in the table? None of the numbers in the list, beyond a certain point, seem to be affecting the first four digits of the subtotals. So, if you were somehow able to add up *all* of the numbers in the infinite list, what do you think the first four digits of the total would be?

Find the first six decimals of the sum of the numbers in our infinite list.

What would you do to find the first ten decimals of the sum of the numbers in our infinite list? (You don't have to actually do it.)

How would you describe the sum of our infinite list of numbers using the concept of "limit"?

Formal Language of Infinite Series.
Using the proper terminology, we will discuss what you have just done. We had a list of numbers (which we know is called a *sequence* of numbers):

$$1, \qquad 1/2^2, \qquad 1/3^3, \qquad 1/4^4, \qquad 1/5^5, \qquad 1/6^6, \qquad \text{etc.}$$

which we call the **TERMS** of the **INFINITE SERIES** and denote it using the Greek letter $\sum$

$$1 + 1/2^2 + 1/3^3 + 1/4^4 + 1/5^5 + 1/6^6 + \ldots = \sum_{k=1}^{\infty} 1/k^k.$$

We tried to find the sum of this infinite series by looking at its **SEQUENCE OF PARTIAL SUMS** (list of subtotals):

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + 1/2^2 \\
S_3 &= 1 + 1/2^2 + 1/3^3 \\
S_4 &= 1 + 1/2^2 + 1/3^3 + 1/4^4 \\
&\vdots \\
S_n &= 1 + 1/2^2 + 1/3^3 + 1/4^4 + \ldots + 1/n^n \\
&\vdots
\end{aligned}$$

We found that the sequence of partial sums S_n seemed to have a **LIMIT** (i.e. the subtotals were stabilizing to a particular value), and that the limit of this sequence of partial sums was the **SUM** of the infinite series:

$$\sum_{k=1}^{\infty} 1/k^k = \lim_{n\to\infty} S_n.$$

When the sequence of partial sums S_n of an infinite series has a limit, the infinite series is said to **CONVERGE**.
When the partial sums S_n do not have a limit, the infinite series is said to **DIVERGE**.
Therefore, in this case, the infinite series $\sum_{k=1}^{\infty} \frac{1}{k^k}$ that we have been examining **CONVERGES**.

$\boxed{\textbf{Exercise}}$

$$\sum_{k=1}^{\infty} (-1)^k$$

Partial sums (fill in the sums):

$S_1 = (-1) =$

$S_2 = (-1) + 1 =$

$S_3 = (-1) + 1 + (-1) =$

$S_4 = (-1) + 1 + (-1) + 1 =$

From your partial sums above, do the partial sums have a limit?
Does the infinite series converge?

THEOREM

If the series $\sum_{k=1}^{\infty} a_k$ is convergent, then $\lim_{k\to\infty} a_k = 0$

We Have A Test For Divergence!

Note the contrapositive statement of this theorem is also rather important to remember:

$$\lim_{k\to\infty} a_k \neq 0 \Rightarrow \sum_{k=1}^{\infty} a_k \text{ is } \textbf{divergent}$$

EXAMPLE 1

$\sum_{k=1}^{\infty} \frac{1}{k}$ (This is called the **HARMONIC SERIES**.)

Partial sums (fill in the sums):

$S_1 = 1 =$

$S_2 = 1 + 1/2 =$

$S_3 = 1 + 1/2 + 1/3 =$

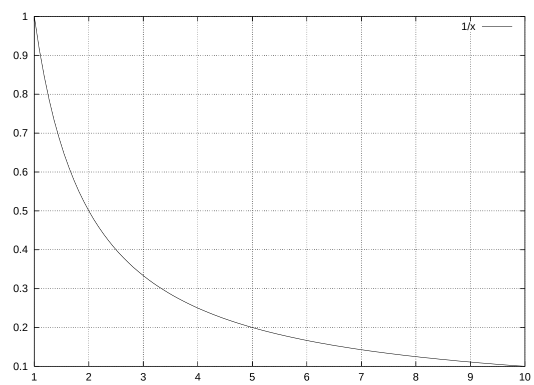
$S_4 = 1 + 1/2 + 1/3 + 1/4 =$

$S_5 = 1 + 1/2 + 1/3 + 1/4 + 1/5 =$

Do you think these partial sums have a limit?

We need to come up with a systematic way of determining the convergence or divergence of an infinite series. Over the next week or so we will learn about **Convergence Tests**.

Let us look at the Left-hand Riemann Sum approximation **L** of the area under the curve $f(x) = 1/x$ from $a = 1$ up to $b = 10$ with $\Delta x = 1$. Sketch this approximation below...



Is **L** an over-estimate or an under-estimate of the exact area under the curve from 1 to 10?

What is the relationship between the Left-hand Riemann Sum $LEFT(10)$, S_{10} and the $\int_1^{10} \frac{1}{x}\,dx$? Write in those relationships (<, > =, etc) below...

$LEFT(10)$ $\qquad$ S_{10} $\qquad$ $\int_1^{10} \frac{1}{x}\,dx$

$\lim_{N\to\infty} LEFT(N)$ $\qquad$ $\lim_{N\to\infty} S_N$ $\qquad$ $\lim_{N\to\infty} \int_1^{N} \frac{1}{x}\,dx$

So, by geometry we can show that $\sum_{k=1}^{\infty} \frac{1}{k}$, the HARMONIC SERIES, ________________

1. INTEGRAL TEST

If $a(k) > 0$ for all k

If $\int_1^\infty a(k)\,dk$ CONVERGES, then $\sum_{k=1}^{\infty} a(k)$ CONVERGES.

If $\int_1^\infty a(k)\,dk$ DIVERGES, then $\sum_{k=1}^{\infty} a(k)$ DIVERGES.

GroupWork

Determine whether the following infinite series CONVERGE or DIVERGE.

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\sum_{k=1}^{\infty} k^2$$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

Connection Between Improper Integrals of the First Kind and Infinite Series

By applying the integral test to the infinite series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ and reviewing the examples above fill in the appropriate condition on p in the RULE below

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \begin{cases} \textbf{CONVERGES} & \text{when } p \\ \textbf{DIVERGES} & \text{when } p \end{cases}$$