Math 120

Calculus 2

Spring 2014

Class 20: Wednesday March 19

Introduction to Infinite Sequences

Warm-Up

(a) What is the next number in this sequence: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$?

(b) Write a formula for the k^{th} number in the sequence: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$?

(c) Is there a "last" number in this sequence? If so, what is it?

Sequences

DEFINITION: Sequence

We define a **sequence** as an infinite list of numbers written in a definite order.

The formal definition of a sequence is that is that is the range (set of outputs) of a function that uses the the natural numbers $\mathbb{N}$ as its domain (set of inputs). We typically write a sequence as $a_1, a_2, a_3, \ldots$, where the a_k are called the **terms of the sequence**) and we refer to the entire series as a whole as $\{a_n\}$ or (sometimes) $\{a_n\}_{n=1}^{\infty}$.

DEFINITION: Limit of a Sequence

A sequence $\{a_n\}$ has a **limit** L if for every $\epsilon > 0$ there exists a corresponding integer N (which depends on ϵ) so that if $n > N$ then $|a_n - L| < \epsilon$.

Basically, this formal definition means that if a sequence has a limit then you can always find a term in the sequence which is as close to the limit value L as you desire.

Notation

When a sequence has a limit L we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$.

Divergent, It's Not Just At The Movies!

If $\lim_{n\to\infty} a_n$ exists then we say the sequence **converges** and call it a **convergent sequence**.

If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n = \pm\infty$ then we say the sequence **diverges** and call it a **divergent sequence**. (Divergent sequences are not dangerous!)

EXAMPLE

Let's Use the horizontal axis as n and the vertical axis as a_n to draw a picture illustrating the convergence of the sequence $a_n = \frac{1}{n}$. What's the limit L of this sequence?

THEOREM

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ where n is a (positive) integer, then the sequence $\{a_n\}$ converges to L, i.e. $\lim_{n\to\infty} a_n = L$.

Exercise

What is the limit of the sequence $a_n = \dfrac{n}{n+1}$? Write out the first four terms of the sequence...

RECALL: L'Hôpital's Rule

If the limit on the left has an indeterminate form (i.e. $\dfrac{0}{0}$, $\dfrac{\pm\infty}{\pm\infty}$ or $\pm\infty \cdot 0$) then it is equal to the limit on the right (if this limit exists)

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to\infty} f'(x)}{\lim\limits_{x\to\infty} g'(x)}$$

When we are working with sequences we often find ourselves dealing with indeterminate forms of the type $\dfrac{\pm\infty}{\pm\infty}$ or $\pm\infty \cdot 0$.

By remembering to use L'Hôpital's rule we can find the limits of a whole bunch of new functions, and we will thus have a nice way to evaluate indeterminate limits.

Special Sequence

Let's analyze the sequence $\{r^n\}$. For what values of r will this sequence converge?

Summarize The Results Here

$$\lim_{n\to\infty} r^n = \begin{cases} \quad , \text{ if} \\ \quad , \text{ if} \\ \quad , \text{ if} \\ \quad , \text{ if} \end{cases}$$

GroupWork

Find the limits of the following sequences:

(a) $a_n = \dfrac{\ln n}{n}$ **(b)** $a_n = \dfrac{e^n}{\ln n}$ **(c)** $a_n = \dfrac{\cos^2(n)}{n^2}$

(d) $a_n = 2 + \dfrac{3n}{n^2+1}$ **(e)** $a_n = \dfrac{4n^2+n+7}{3n^2+1}$ **(d)** $a_n = (-1)^n$

THEOREM

If $\lim\limits_{x\to\infty} a_n = L$ and the function $f(x)$ is continuous at L then $\lim\limits_{n\to\infty} f(a_n) = f(L)$

DEFINITION: monotonic sequence

A sequence $\{a_n\}$ is said to be **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$. In other words $a_1 < a_2 < a_3 < \ldots$. A sequence $\{a_n\}$ is said to be **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. In other words $a_1 > a_2 > a_3 > \ldots$.
If a sequence that is decreasing or increasing is called **monotonic**.

EXAMPLE

Adapted from **Stewart, p. 435, Exercise 37**. Show that $a_n = \dfrac{1}{2n+3}$ is monotonic.

DEFINITION: bounded sequence

A sequence $\{a_n\}$ is said to be **bounded above** if there exists a number M so that $a_n \leq M$ for all $n \geq 1$. A sequence $\{a_n\}$ is said to be **bounded below** if there exists a number m so that $a_n \geq m$ for all $n \geq 1$. If a sequence is bounded above and bounded below it is said to be **bounded**.

EXAMPLE

Adapted from **Stewart, p. 435, Exercise 37**. Show that $a_n = \dfrac{1}{2n+3}$ is bounded.

THEOREM

Every bounded, monotonic series is convergent!