Class 18: Monday March 17

Applications of Integration: Finding Lengths of Curves

`Warm-Up`

(a) What's the circumference of a circle of radius R?

(b) What is distance between two points as (x_1, y_1) and (x_2, y_2)?

FORMULA: **Length of a Curved Segment**

The length L of a curved segment represented by the graph $y = f(x)$ over the interval $a \leq x \leq b$ is given by

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

(NOTE: This formula assumes that the derivative $f'(x)$ exists and is continuous on the interval $a \leq x \leq b$.)

The arc length formula can also be written using Leibniz' notation for the derivative:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

EXAMPLE

Exercise #18, page 391 of Stewart Find the length of $y = 1 - e^{-x}$ between $0 \leq x \leq 2$.

The Arc Length Function

Given a function $f(t)$, we can define a function $s(x)$ that measures the distance along the curve from a starting point **P** $(a, (f(a))$ to another point **Q** $(x, f(x))$ as

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\ dt$$

By the Fundamental Theorem of Calculus Part 2, we can write

$$\begin{aligned} \frac{ds}{dx} &= \sqrt{1 + [f'(x)]^2} \\ &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \end{aligned}$$

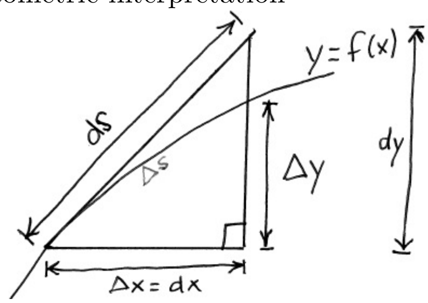
This means we can get the famous expression that relates the change in arc length (ds) to the change in input (dx) and change in output (dy) of a function $y = f(x)$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

or that

$$(ds)^2 = (dx)^2 + (dy)^2$$

This equation has an interesting geometric interpretation



Exercise

Stewart, pg. 392, Exercise 29. Find the arclength function for the curve $y = 2x^{3/2}$ with starting point $P_0(1,2)$.