Class 13: Monday February 24

Improper Integrals, Part 2

Warm-Up

(a) An improper integral can either ________________________ or ________________________.

DEFINITION

An integral is said to be an **Improper Integral** if one of the limits of integration is infinite or if the integrand becomes unbounded at some point in the interval of integration.

There are two types of improper integrals:

1. **Improper Integrals of the First Kind**

e.g. $\int_a^\infty f(x)\,dx$, $\int_{-\infty}^b f(x)\,dx$ or $\int_{-\infty}^\infty f(x)\,dx$

2. **Improper Integrals of the Second Kind**

$\int_a^b f(x)\,dx$, where either $\lim_{x\to a^+} f(x) = \infty$ or $\lim_{x\to b^-} f(x) = \infty$ or $\lim_{x\to c} f(x) = \infty$ when $a \le c \le b$

Evaluation of Improper Integrals

To evaluate an improper integral of the first kind one rewrites it as:

$$\begin{aligned}\int_a^\infty f(x)\,dx &= \lim_{b\to\infty}\int_a^b f(x)\,dx\\ \int_{-\infty}^b f(x)\,dx &= \lim_{a\to-\infty}\int_a^b f(x)\,dx\end{aligned}$$

If the limit exists, then the improper integral is said to *CONVERGE*.
If the limit is unbounded, the improper integral is said to *DIVERGE*.

To evaluate an improper integral of the second kind (assuming $f(x)$ is unbounded at some point $x = c$ where $a \le c \le b$):

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\\ &= \lim_{b\to c^-}\int_a^b f(x)\,dx \quad + \quad \lim_{a\to c^+}\int_a^b f(x)\,dx\end{aligned}$$

Similarly, if the limit exists (and is finite), then the integral *CONVERGES*.
If the limit does not exist (or is unbounded) then the improper integral *DIVERGES*.

Exercise

How many possible outcomes are there with improper integrals? Write down an example of an integral associated with each type of outcome.

GroupWork

Evaluate the following integrals. If the integral is improper, say what KIND of improper integral it is and determine whether it CONVERGES or DIVERGES. Look for patterns, and try and discover the rules found on the next page.

1. $\displaystyle\int_2^\infty \frac{1}{x^{1.000001}}\,dx$

2. $\displaystyle\int_5^\infty \frac{1}{s^{0.99999999}}\,ds$

3. $\displaystyle\int_0^2 \frac{1}{t^4}\,dt$

4. $\displaystyle\int_1^5 \frac{1}{t-2}\,dt$

5. $\displaystyle\int_0^8 \frac{1}{\sqrt[3]{r}}\,dr$

6. $\displaystyle\int_0^3 x^{6/5}\,dx$

7. $\displaystyle\int_1^\infty e^{-2s}\,ds$

8. $\displaystyle\int_{-\infty}^1 e^{4r}\,dr$

Important Limits To Know

Let's summarize our knowledge of improper integrals and limits.

$$\int_a^\infty \frac{dx}{x^p} = \begin{cases} \underline{\hspace{4cm}} & \text{when } p \leq 1 \\ \underline{\hspace{4cm}} & \text{when } p > 1 \end{cases}$$

$$\int_0^b \frac{dx}{x^p} = \begin{cases} \underline{\hspace{4cm}} & \text{when } p \geq 1 \\ \underline{\hspace{4cm}} & \text{when } p < 1 \end{cases}$$

$$\lim_{b\to\infty} b^p = \begin{cases} \underline{\hspace{4cm}} & \text{when } p < 0 \\ \underline{\hspace{4cm}} & \text{when } p > 0 \end{cases}$$

$$\lim_{x\to\infty} e^{kx} = \begin{cases} \underline{\hspace{4cm}} & \text{when } k < 0 \\ \underline{\hspace{4cm}} & \text{when } k > 0 \end{cases}$$

The Comparison Test For Improper Integrals

With improper integrals, we often just want to know whether they CONVERGE or DIVERGE, and we don't have to evaluate them directly. Under certain conditions, we can use the **Comparison Theorem** to determine the convergence or divergence of an improper integral without evaluating it.

THEOREM

Given two continuous functions $f(x)$ and $g(x)$ such that $f(x) \geq g(x) \geq 0$ for $x \geq a$ and associated improper integrals $\mathcal{I} = \int_a^\infty f(x)\,dx$ and $\mathcal{J} = \int_a^\infty g(x)\,dx$.

(a) If $\mathcal{I}$ is convergent then $\mathcal{J}$ must be convergent

(b) If $\mathcal{J}$ is divergent, then $\mathcal{I}$ must be divergent.

The basic idea is if that I can show that the improper integral I am given represents an area ____________________ than an improper integral I know converges, then my integral must also CONVERGE.

Similarly, if I can show that the integral I have has an integrand which is always ____________________ than the integrand of an improper integral I know diverges, then my integral must also DIVERGE.

EXAMPLE

Show that $\int_1^\infty e^{-x^2}\,dx$ is a convergent improper integral.

Exercise

Show that $\int_2^\infty \frac{1}{\ln(x)}\,dx$ is a divergent improper integral.

Trick Question

Look at the evaluation of the following integral. Is there anything wrong with this calculation? How would you do it differently? $I = \int_{-1}^{2} \frac{1}{x^2}\,dx = \left.\frac{-1}{x}\right|_{-1}^{2} = \frac{-1}{2} - \frac{-1}{-1} = \frac{-1}{2} - 1 = \frac{-1}{2}$