Math 120 | **CALCULUS 2** | Spring 2014

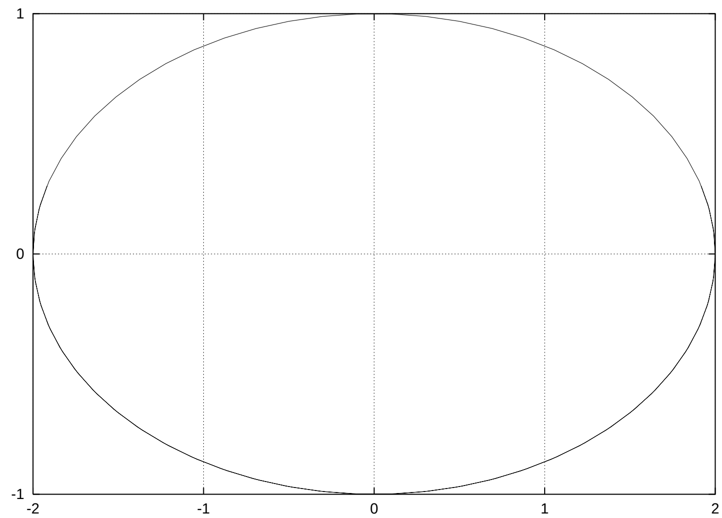
Class 1: Wednesday January 22

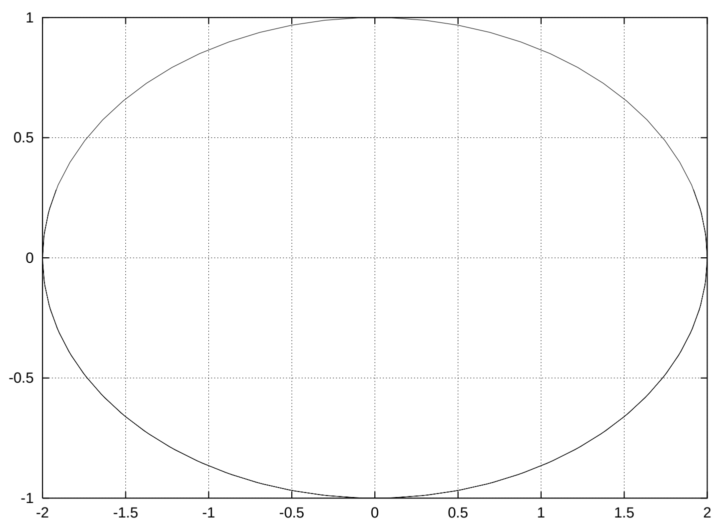
Introduction to Accumulation, Approximating Area and Riemann Sums

Accumulation of Area (Over-estimates and Under-estimates)

In groups of two or three, try to estimate the area of the ellipse $x^2 + 4y^2 = 4$ below using the grids provided.

- First, underestimate the area **A1** using the larger (whole units) grid;
- second, overestimate the area **A2** using the larger (whole units) grid;
- third, underestimate the area **A3** using the smaller grid and
- fourth, overestimate the area **A4** using the smaller grid.





$$\quad \leq \quad \leq \text{ EXACT AREA } \leq \quad \leq$$

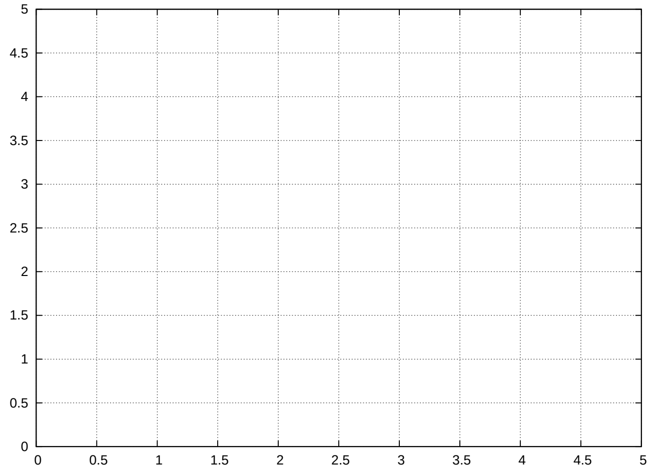
DISCUSS

QUESTION: How could these estimates be improved? What procedure or tools would you use improve your estimates of the area of the ellipse?

ANSWER:

Using Riemann Sums To Estimate Areas

Suppose we want to find the area A of the region described below:
$0 \leq x \leq 2$, above the x-axis, and below $f(x) = x^2 + 1$.



Sketch and shade the region described above.

In the shaded region draw two rectangles of equal width, neither of which goes above the curve $f(x) = x^2 + 1$.
The width of each rectangle is __________.
The heights of the two rectangles are __________ and __________.
The sum of the areas of the two rectangles is __________. This is your approximation $\tilde{A}_2$.
This means your approximation $\tilde{A}_2$ of the area A will be an ________________________.

Find a better estimate $\tilde{A}$ for the area of the shaded region A
by repeating the above procedure, but this time with FOUR rectangles. Let's call it $\tilde{A}_4$.

Repeat with 100 rectangles to find $\tilde{A}_{100}$! Just write the appropriate expression, but do not evaluate it. This expression is called a **Riemann Sum**.

Riemann Sum

The general form of a Riemann Sum for a given function $f(x)$ evaluated at points separated by Δx is

$$\sum_{k=1}^{N} f(x_k)\Delta x$$

Now repeat your approximation with N rectangles, instead of 100.

Can you see how this new expression $\tilde{A}_N$ for area A **also** has the form of a Riemann Sum?
The larger N is, the more __________ the estimate of $\tilde{A}_N$ is to area A. So, the EXACT value of the area can be written down as:

Riemann Sums Using Other Sample Points

On this worksheet, we have been doing **left-hand** Riemann Sums. This means:

In each "sub-interval" we used the _______ endpoint to calculate the height of each rectangle.

There was no special reason to use LEFT-hand sums. We could just as well use RIGHT-hand Riemann Sums, and everything would still work the same (except in this example we'd happen to get over-estimates). In fact, as long as you evaluate the function at SOME POINT in each sub-interval and multiply by the size of that sub-interval and add up your products we still get a Riemann Sum.

IF, in the LIMIT, the Riemann sums give the same exact number regardless of sample point THEN this number is called the value of **the definite integral**.

GROUPWORK

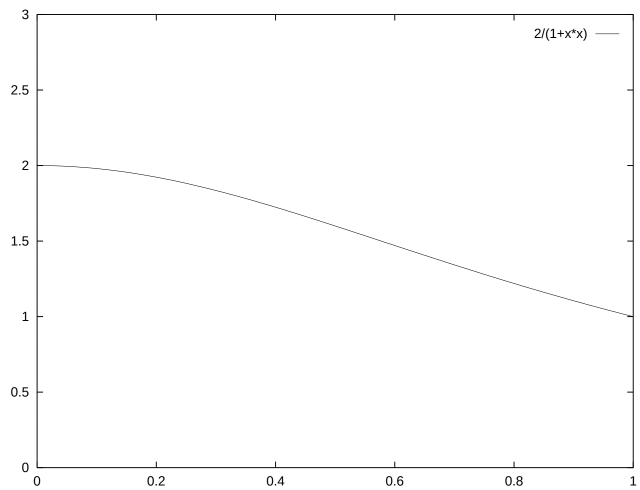
Compute the right-hand Riemann Sum approximation for the area A, using two rectangles, and then again using four rectangles. Let's call these R_2 and R_4.

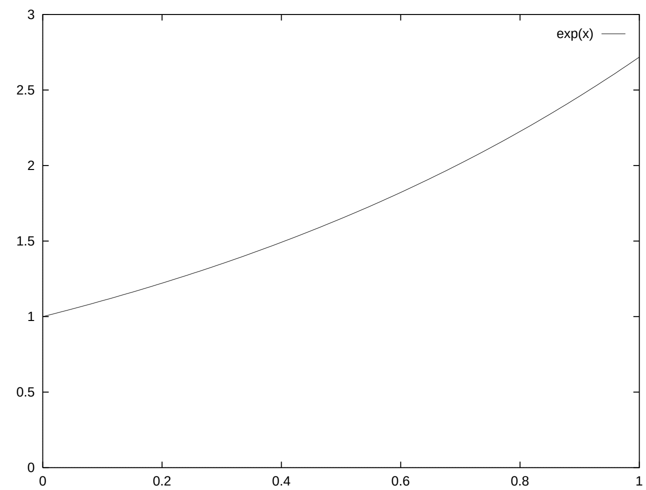
Let's also use the midpoint Riemann Sum to obtain approximations for the area A, using four rectangles. Call this M_4

What do you notice about the relative sizes of $\tilde{A}_4$, R_4 and M_4? (We'll explore these relationships further in a number of cases in the lab in future.)

Estimating The Error In Riemann Sum Approximations

Let's look at another example of accumulation to get an estimate for the area of an arbitrary shape. We will also try to get a sense of "how good" our estimate is.





We will use *Riemann Sums* to approximate the area "under the curves" in the figures. We will use *Error Stacks* to bound the error of our approximation.

<u>Riemann Sums</u> and <u>Error Stacks</u>

Error Bound for Riemann Sum Area Estimates

We can show that we know that the error we make in using Riemann Sums to approximate the area under a curve which is monotonic increasing or decreasing on the interval of interest is bounded.
In other words,

$$|\text{EXACT AREA} - \text{APPROXIMATE AREA}| \leq |f(b) - f(a)|\Delta x$$