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Lab #12
Math 120 Lab
Thursday
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Fourier Series Approximations

§1 Fourier Series

Suppose that $f(x)$ is a periodic function with period 2π. If we want to approximate this function with a trigonometric polynomial of degree n,

$$F_n(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \cdots + a_n\cos(nx) + b_n\sin(nx)$$

then the "best" coefficients to use are the following **Fourier coefficients**:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\ dx \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\ dx \end{aligned}$$

where $k \geq 1$.

Whereas a Taylor Series attempts to approximate a function locally about the point where the expansion is taken, a Fourier series attempts to approximate a periodic function over its entire domain. That is, a Taylor series approximates a function pointwise and a Fourier series approximates a function globally.

Error in Using Fourier Series

The error in approximating a 2π-periodic function $f(x)$ by a Fourier polynomial $F_n(x)$ is given by the following integral:

$$E_n = \int_{-\pi}^{\pi} |f(x) - F_n(x)|^2\ dx.$$

That is, $F_n(x)$ is a "good" approximation of $f(x)$ if E_n is small. Notice that E_n does not depend on x. E_n gives a measure of how well $F_n(x)$ approximates $f(x)$ over all of $[-\pi, \pi]$, not just at a single point as our error bound for Taylor polynomials did. The above Fourier coefficients are the "best" coefficients in the sense that those coefficients make E_n as small as possible.

Part 1: Computing Fourier series

Conside the sawtooth wave of period 2π given by $f(x) = x$ when $-\pi \leq x \leq \pi$. Give a sketch of the sawtooth wave from $-4\pi \leq x \leq 4\pi$

Find the first three Fourier polynomials, $F_1(x), F_2(x)$, and $F_3(x)$.

(Hint: a_0 and the rest of the a_k's are easy to determine with a bit of thought. Focus on the graph of the function and think about the area.)

Now write down a general formula for the Fourier coefficients a_k and b_k. You may want to break the integral up into two parts to compute b_k. You can use Wolfram's integration capabilities to check your answer.