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Lab #5
Math 120 Lab
Thursday
February 20, 2014
Prof. Ron Buckmire

Simpson's Rule

Introduction

The integral of a function on an interval is defined to be the limit of the Riemann sums for the function. In some cases, it is possible to use the Fundamental Theorem of Calculus to bypass the definition and use antiderivatives to evaluate integrals. More often, finding antiderivatives is difficult or even impossible, and numerical methods are used instead. This lab illustrates one of the most-often used methods for evaluating definite integrals. Calculators such as the TI-85 and apps like `Wolfram | Alpha` use a variant of the numerical method known as "Simpson's Rule."

We will be using the classic **TrueBasic** programs `RIEMANN`, `SIMPSON` and `SUMCOMP` during this lab. These programs are found on the Occidental Network drive, at `S:\Math Courses\Math120\Spring2014\Lab`

Rectangular Approximations

We have seen the following three methods of approximating an integral using a Riemann sum.

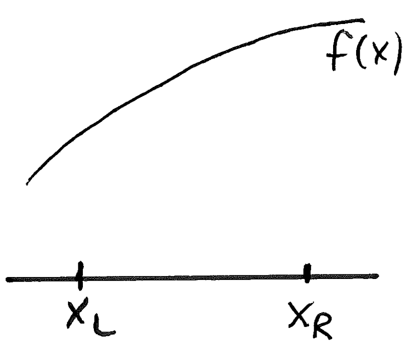
Left-Endpoint: $$\int_a^b f(x)\ dx \approx L = \sum_{k=1}^{N} f(x_k)\Delta x,$$ using $x_k = a + (k-1)\Delta x$

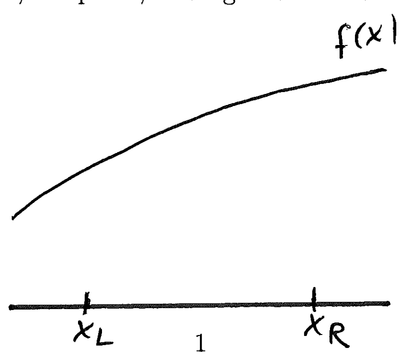
Midpoint: $$\int_a^b f(x)\ dx \approx M = \sum_{k=1}^{N} f(x_k)\Delta x,$$ using $x_k = a + (k-\frac{1}{2})\Delta x$

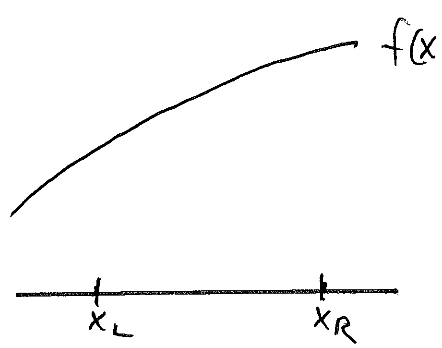
Right-Endpoint: $$\int_a^b f(x)\, dx \approx R = \sum_{k=1}^{N} f(x_k)\Delta x,$$ using $x_k = a + (k)\Delta x$

Each of them can be viewed as **rectangular** approximations on each subinterval.

The three sketches below show the graph of a positive function on an interval $x_L \leq x \leq x_R$ of width Δx. Sketch in the rectangles corresponding to the left, midpoint, and right estimates. Write out these estimates as a formula.

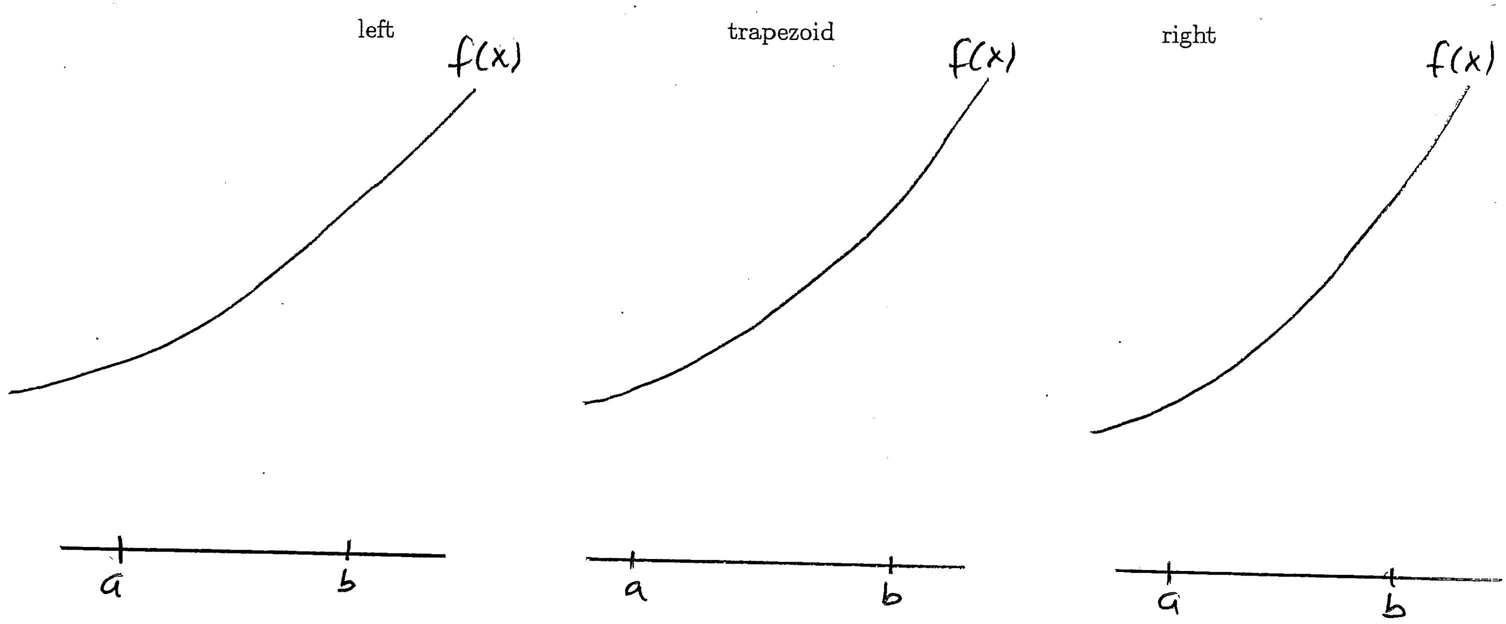






Approximations using trapezoids

1. **"The" trapezoid approximation.** On the sketches below, fill in the rectangles corresponding to the left and right estimates. On the sketch labeled "trapezoid," sketch in the slant line joining the points at the ends of the interval.



a. Calculate the area of the trapezoid in this approximation.

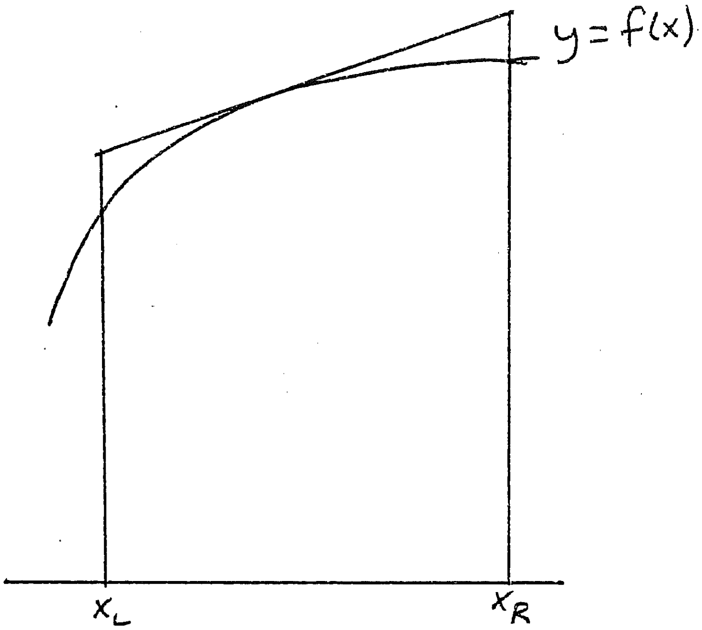
b. Calculate the average of the left and right approximations. You should arrive at the same formula as above.

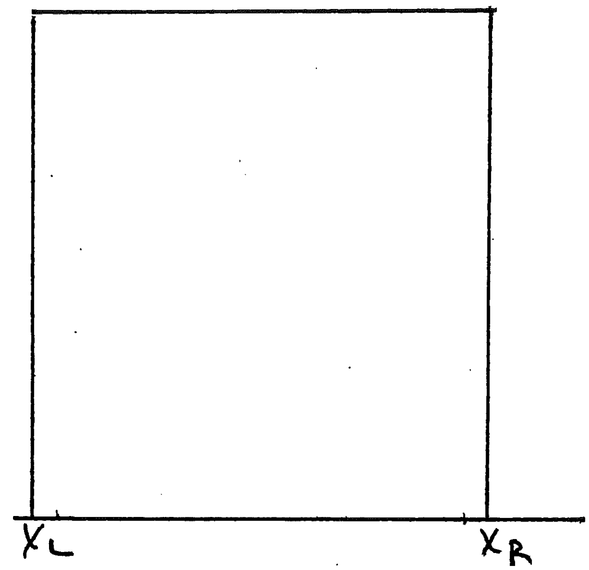
The approximation to the integral formed by averaging the left and right approximations is called **the Trapezoid approximation.**

$$\textbf{Trapezoid:} \qquad \int_a^b f(x)\ dx \approx T \ = \tfrac{1}{2}(L+R)$$

Most calculus books include only this one approximation by trapezoids: it is customary to call it "the" trapezoid approximation.

2. **The midpoint estimate *as* a trapezoid.** The sketch below shows the graph of a function and an approximation to its integral using a trapezoid with slant side *tangent* to the curve at the midpoint.



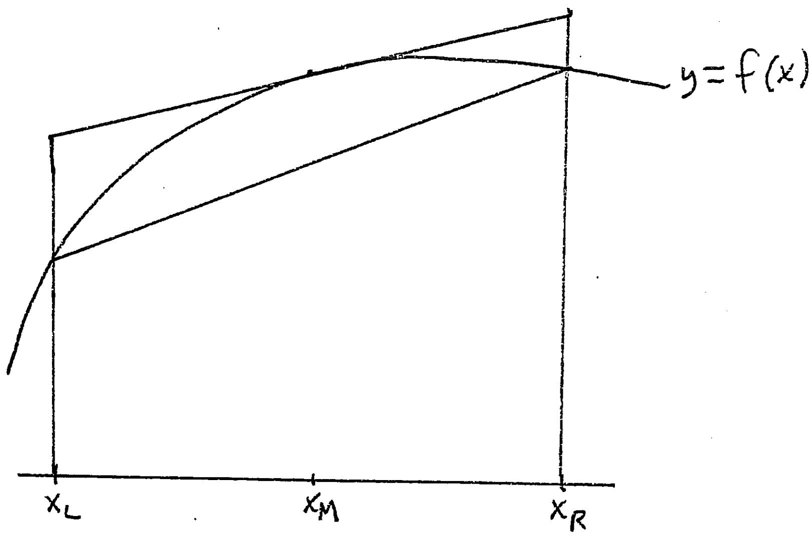


(a) What is the average height of the trapezoid in the diagram?

(b) Using your previous answer, what is the area of the trapezoid in the figure?

(c) Why is approximation by this trapezoid the same as the midpoint estimate?

3. **Comparing the trapezoid and midpoint approximations.** The sketch below shows the graph of $y = f(x)$ over a small interval. Added to the sketch are two lines: the line joining the endpoints of the graph on this interval and the line tangent to the graph at the midpoint.



(a) For this example, is the trapezoid approximation an over- or under-estimate? Shade in the area corresponding to the amount of the error.

(b) Is the midpoint approximation too much or too little? Shade in the area corresponding to the amount of the error. (Use a different shading method or a different color.)

(c) The picture might mislead someont to believe that the two slant sides of the trapezoids are parallel. Sketch an example of a monotonic concave-down function where the two slant sides are *not* parallel.

(d) Sometimes "the" trapezoid method will be an overestimate and sometimes an underestimate. What feature of the graph can be used to determine whether the trapezoid method gives an overestimate or an underestimate?

(e) In this example, which is larger: the amount of the trapezoid error or the amount of the midpoint error?

Numerical comparisons

The next few exercises will compare the accuracy of approximating integrals using the four methods described above. In order to test the accuracy, we will use examples where the exact value of the integral can be calculated independently. (Exact evaluation of integrals is sometimes not practical; in this case, numerical methods are used as the *only* way to estimate the value of an integral.)

4. A polynomial example.

   a. Use the Fundamental Theorem of Calculus to give the exact value of the following integral:

   $$\int_0^2 x^5 + 1 \, dx =$$

   b. Use the program `RIEMANN` to fill in the second line of the following table of estimates to the integral you just computed. Keep at least five places beyond the decimal point.

N	L	R	T	M
5				
20				
50				
100				

   Which of the methods is most accurate for this integral?

5. **Comparing errors numerically.** Each estimate differs from the exact value of the integral by some amount: this amount is called an "error" in approximation. If the amount of error is small, the estimate is a good one. These appproximation errors are not "mistakes."

   The program `SUMCOMP` makes it easier to compare the errors for each of the four methods. Examine the listing of this program before running it. (Really!) You need to specify the formula for the function, the interval, and *the exact value of the integral.* The program prompts you for the number of subintervals and then reports the computed approximations and the "error" for each method.

   The errors are computed as follows, using I as the exact value of the integral:

   $$\begin{aligned} I - L &= \text{error for the left-endpoint method} \\ I - R &= \text{error for the right-endpoint method} \\ I - T &= \text{error for the trapezoid method} \\ I - M &= \text{error for the midpoint method} \end{aligned}$$

   (a) To verify the program is working correctly, use it to approximate the same integral as before. Choose N = 50 subdivisions. (Note that this program prompts you for N when you run it.) How do your results compare? Now fill in the rest of the table above.

   (b) Some of the errors are positive, some negative. Which sign corresponds to an overestimate, which to an underestimate?

6. **Patterns in the errors.** By studying the errors in approximation in the trapezoid and midpoint methods, we can develop an even better approximation method.

(a) Use SUMCOMP to complete the following table of errors. $I = \int_0^2 x^2\, dx =$ ________________________

(Fill in the exact value.)

N	$I-T$	$I-M$
5		
20		
50		

(b) Now calculate a table of errors for two integrals of your choice: one a polynomial of degree greater than 2 and one non-polynomial. $I = \int \qquad dx =$ ________________________

N	$I-T$	$I-M$
5		
20		
50		

$I = \int \qquad dx =$ ________________________

N	$I-T$	$I-M$
5		
20		
50		

(c) How do the *signs* of the errors compare?

(d) How do the *magnitudes* of the errors compare?

(e) Calculate the ratio $\frac{I-T}{I-M}$ for some of the examples above. What pattern do you see?

(f) Use the previous result to explain why $\frac{2}{3}M + \frac{1}{3}T$ should be a better approximation to I than either M or T.

Simpson's Rule

The estimate $\frac{2}{3}M + \frac{1}{3}T$ is called "Simpson's Rule."

$$\textbf{Simpson:} \qquad \int_a^b f(x)\,dx \approx S \quad = \frac{2}{3}M + \frac{1}{3}T$$

The program `SIMPSON` uses this method to calculate an approximation of an integral. Use this program to calculate successive approximations to at least some of the following integrals. (Complete all of them, after the lab session.) Successively choose values of N until you have stable values to three decimal places. (We have not given an analysis of when the Simpson approximation is an over- or under-estimate.)

1. $I = \int_0^2 -4x + 1\,dx =$ ______________

N	Simpson Approximation

2. $I = \int_0^2 3x^2 - x + 1\,dx =$ ______________

N	Simpson Approximation

3. $I = \int_0^2 x^3 + 2x^2\,dx =$ ______________

N	Simpson Approximation

4. $I = \int_0^2 x^4 - 2x^2\,dx =$ ______________

N	Simpson Approximation

5. $I = \int_0^3 x\ln(x^2+1)\,dx =$ ______________

N	Simpson Approximation

6. $I = \int_0^2 \sqrt{4-x^2}\,dx =$ ______________

N	Simpson Approximation

7. $I = \int_0^3 \frac{1}{1+x^5}\,dx =$ ______________

N	Simpson Approximation

How do the answers obtained using Simpson's Rule compare to the value you find when you use `Wolfram | Alpha`